Geometric Insight into Return Mapping Plasticity Algorithms

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ABSTRACT

Return mapping algorithms are probably the most popular means of numerically solving conventional plasticity equations. However, numerous unfortunate misconceptions regarding the geometrical aspects of stress space and return algorithms will be addressed. The basic tenets of these techniques are here rigorously justified and interpreted geometrically in 6D stress space. For any return algorithm, the first step is to tentatively assume elastic behavior throughout a given time step. If the resulting "trial" stress is forbidden (i.e. if it violates the yield condition), then the tentative assumption of elastic response is rejected. Even when it is found to violate the yield condition, the trial stress is nevertheless useful because it can then be projected back to the plastic yield surface to give the updated stress. Return algorithms are often wrongly regarded as numerical "tricks" because they appear to be ad hoc means of keeping the stress on the yield surface. It is natural to inquire whether other approaches might be more accurate for the same computational cost, but it is shown here that return methods are rigorously justifiable and appear to correspond to optimal numerical accuracy and efficiency. It is shown that issues such as plastic stability, dissipation, and convexity dictate appropriate choices for the quantities that are presumed known in the derivation of return algorithms; it is not the return algorithm *per se* that addresses such physical concerns. Code users and even many model developers often seem to believe (in error) that the method used to return the stress to the yield surface becomes inconsequential as the time step is reduced. However, it is proved that the correct return direction is dictated by the governing equations, and any other return direction will converge, but not to the correct solution of the governing equations. Furthermore, the correct return direction is *not aligned with the plastic strain rate* except under certain conditions. Consequently, *normality of the plastic strain rate does not necessarily correspond to normality of the return direction*, and vice versa. These claims are proved first in the context of stationary yield surfaces and then generalized to permit hardening or softening. This technical note is intended to provide nothing more than geometrical insight into known results.

The direction used to return to the yield surface really matters!

Before launching into details of the theory of return algorithms, a motivational discussion is essential. Plasticity algorithms are just one instance of a larger class of material models in which the behavior of a material is presumed to change markedly once the stress reaches a critical value. In the field of plasticity, the set of stresses marking the transition boundary is called the yield surface. For non-hardening plasticity, the stress must be constrained to always fall either within the yield surface or on the yield surface. Typically, numerical solutions of the plasticity equations tentatively assume that the entire time increment is purely elastic. If the predicted "trial" updated stress is found to fall outside the yield surface, then the numerical algorithm recognizes that the tentative assumption of elasticity must have been wrong. At

that point, classical return algorithms assert that the correct updated stress can be obtained by simply projecting the inadmissible trial stress back to the yield surface. One goal of this report is to rigorously justify this approach.

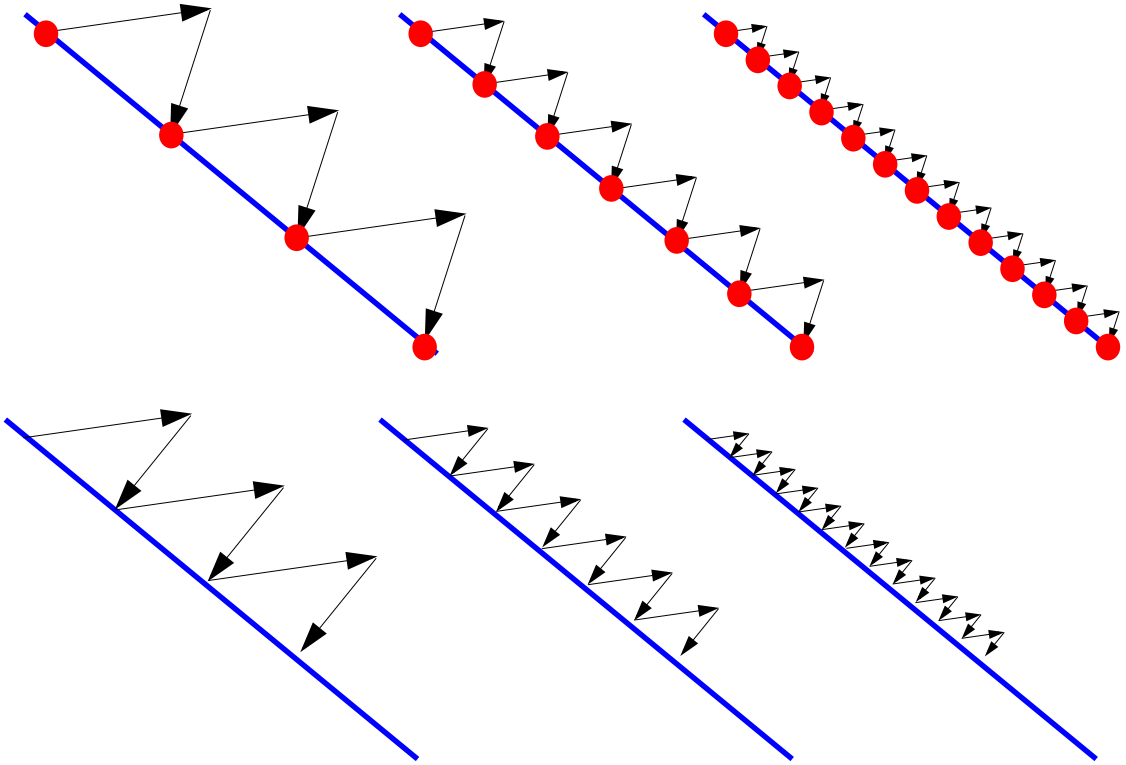

Figure 1. An illustration of return algorithms. The stress state alternates between the trial stress (lying outside the yield surface) and the final predicted stress (lying on the yield surface). The dots in the top set of figures show how the predicted stress moves along the yield surface with successive time steps when the projection back to the yield surface is oblique. The left side shows the result of a large time step and the right side shows the result of a refined time step. Because this illustration uses a flat yield surface, the final results are identical for both coarse and large steps (infinite convergence rate). The bottom set of figures shows the same solution procedure using a different projection direction. Even though both procedures have infinite convergence rates, their final solutions for the stress state differ. Consequently, even though both procedures converge, at least one of them must be converging to the *wrong* result.

Intuitively, it's easy to see that, if the stress state at the beginning of a time interval lies on the yield surface, then the prediction for the projected updated stress at the end of the interval will be the same in the limit as the time step goes to zero. Unfortunately, many researchers wrongly conclude that the direction used to project back to the yield surface is therefore inconsequential. Fig. 1 shows a sketch that illustrates the fallacy of this assertion by using a counterexample of a flat yield surface with a constant applied strain rate. The top three drawings in that figure show how the predicted stress moves along the yield surface for successively smaller time steps, with the total number of time steps being increased appropriately to ensure that the problem end time

is the same in all three cases. The bottom three drawings show how the stress would progress for a *different* return direction. The two approaches clearly predict different results for the final updated stress. In other words, even though both approaches predict *identical* paths in stress space, these paths are traced at *different rates*. Consequently, even though both solutions are converging, at least one of them is converging to the *wrong result*. The erroneous solution must be solving the governing differential equations incorrectly because it is predicting the wrong result for how stress varies with *time*. If the prediction of stress through *time* is wrong, then any finite element code that uses the model will be using the right stresses at the wrong time, which would lead to erroneous solutions to the time-dependent field equations.

So what is the *right* direction to return to the yield surface? Many researchers believe that the direction to return to the yield surface should be parallel to the direction of the plastic strain rate. While this is true in some cases, this report will show that it is not true in general — especially not if the yield surface is pressure sensitive (as is the case, for example, with porous media). Researchers operating under the false assumption that the return direction should be parallel to the plastic strain rate often end up wrongly concluding that they need to use a non-associated (i.e., non-normal) plastic flow direction in order to match observed data.

As will be discussed in this report, another source for erroneous conclusions that a flow law must be non-associated arises when researchers plot the yield function in the space of equivalent shear stress versus pressure. This 2D stress space is *not* isomorphic to 6D tensor stress space, and therefore the angle between the yield surface and the return direction must be different in the two spaces. A “nearest point” return direction (i.e., one for which the return direction is normal to the yield surface) in 6D stress space will appear to be oblique in the 2D space of equivalent shear stress versus pressure, which leads some researchers to wrongly assess the obliqueness (or lack of obliqueness) of their plastic strain rate.

A final source of “phantom obliqueness” arises when the elastic response is coupled to the plastic response, as is the case for porous plasticity models that permit the elastic moduli to stiffen in response to plastic pore collapse. As will be discussed in this report, elastic plastic coupling results in a change in the uniquely required return direction. For porous plasticity models, this change in return direction is such that the isotropic part of the return direction must be smaller than it would be if no elastic-plastic coupling were present. Again, this result is sometimes wrongly interpreted by researchers as evidence that a normal plastic flow direction over-predicts the plastic dilatation.

Notation

The geometrical interpretation of plastic return algorithms is most apparent when presented in symbolic Gibbs notation (also known as "direct" notation). Throughout this document, the number of underlines beneath a symbol indicates the tensorial order of that variable. Hence, for example, s would be a scalar, $\underset{\sim}{u}$ a vector, $\underset{\approx}{A}$ a second-order tensor, and $\underset{\approx\approx}{E}$ a fourth-order tensor. Unless otherwise indicated, the term "tensor" will be taken to mean "second-order tensor."

Understanding radial and oblique return algorithms *demands* a prerequisite understanding that second-order tensors are themselves first-order vectors in 9-dimensional space. The set of all *symmetric* tensors is a 6-dimensional vector subspace, which (in plasticity literature) is misleadingly called "stress space" even though it is well-defined for all other symmetric tensors such as strain. The concept of a "subspace of 9-D tensor space" is analogous to a "plane" that passes through the origin in ordinary 3-D space. Any linear combination of vectors in such a plane is itself in the plane.* Mathematically, a set forms a *subspace* if any linear combination of members of that set is itself a member of the set. Symmetric tensors form a subspace because any linear combination of symmetric tensors is itself symmetric. The set of all orthogonal tensors is *not* a subspace because the sum of two orthogonal tensors is not generally orthogonal.

Any vector operation that is defined for ordinary vectors in 3D space has an analogous operation that is defined for tensors. The geometrical interpretations are identical. For ordinary vectors in 3D space, the single "dot" product between two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, is defined

$$\underset{\sim}{a} \bullet \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i . \tag{1}$$

Note that components of $\underset{\sim}{a}$ are simply multiplied by corresponding components of $\underset{\sim}{b}$. Visualizing second-order tensors as nine-dimensional vectors, the analogous *tensor* inner product between two tensors, $\underset{\approx}{A}$ and $\underset{\approx}{B}$, is defined

$$\begin{aligned} \underset{\approx}{A} : \underset{\approx}{B} &= A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} \\ &\quad + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} \\ &\quad + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}B_{ij} . \end{aligned} \tag{2}$$

*This property is why we added the proviso that the plane must pass through the origin.

The magnitude of a 3D vector $\underset{\sim}{a}$ is defined $\sqrt{\underset{\sim}{a} \bullet \underset{\sim}{a}}$. Similarly, the magnitude of a tensor $\underset{\approx}{A}$ is defined $\sqrt{\underset{\approx}{A} : \underset{\approx}{A}}$. Two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, are *perpendicular* if $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$. Similarly, two tensors, $\underset{\approx}{A}$ and $\underset{\approx}{B}$, are said to be perpendicular if $\underset{\approx}{A} : \underset{\approx}{B} = 0$. When we say that a tensor $\underset{\approx}{A}$ is in the "direction" of some other tensor $\underset{\approx}{B}$, we mean that they are "proportional." In other words, $\underset{\approx}{A} = \gamma \underset{\approx}{B}$ for some scalar multiplier γ. Plasticity interpretations draw heavily on the concept that tensor proportionality is just like vector coaxiality.

Incidentally, it's important to apply the full nine-dimensional tensor inner product formula even when working in six-dimensional symmetric tensor space. If $\underset{\approx}{A}$ and $\underset{\approx}{B}$ happen to be symmetric, then Eq. (2) reduces to

$$\underset{\approx}{A} : \underset{\approx}{B} = A_{11}B_{11} + A_{22}B_{22} + A_{33}B_{33} + \quad 2(A_{23}B_{23} + A_{31}B_{31} + A_{12}B_{12}) \tag{3}$$

Note that the off-diagonal components contribute twice. Appendix __ explains why the tensor-vector analog only *seems* to be breaking down in this case, and that appendix shows how to recover the intuitive analog by a change of basis. The bottom line is that the operation $\underset{\approx}{A} : \underset{\approx}{B}$ remains geometrically analogous to the vector dot product even in symmetric tensor space, even though the *algebraic* definition might seem to be different (see Appendix __).

In 3D space, the operation $\underset{\sim}{y} = \underset{\approx}{A} \bullet \underset{\sim}{x}$ represents a linear transformation from the vector $\underset{\sim}{x}$ to a new vector $\underset{\sim}{y}$. The indicial form of this operation is

$$y_i = \sum_{j=1}^{3} A_{ij} x_j . \tag{4}$$

The summation occurs over every component of $\underset{\sim}{x}$. Analogously, the operation $\underset{\approx}{Y} = \underset{\approx\approx}{E} : \underset{\approx}{X}$ represents a linear transformation from the tensor $\underset{\approx}{X}$ to a new tensor $\underset{\approx}{Y}$. The indicial form of this operation is

$$Y_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} E_{ijkl} X_{kl} . \tag{5}$$

Analogous to Eq. (4), the summation occurs over every component of $\underset{\approx}{X}$.

Linear tensor transformations of the form $\underset{\approx}{Y} = \underset{\approx\approx}{E} : \underset{\approx}{X}$ play a pivotal role in material modeling. After all, what is a material model? It is a rule by which you start with one tensor (e.g., the strain) as input and compute some other tensor (e.g. stress) as output of the model. Thus, a constitutive model is a transformation taking tensors to tensors. By why should *linear* transformations be so important? Any experimentalist will tell you that most materials behave in a *nonlinear* fashion. The answer is that a nonlinear function becomes *linear* in rate form. Consider, for example, the most general expression for nonlinear elasticity: namely, the stress $\underset{\approx}{\sigma}$ is presumed to be a nonlinear function of the strain $\underset{\approx}{\varepsilon}$:

$$\underset{\approx}{\sigma} = f(\underset{\approx}{\varepsilon}) \tag{6}$$

Applying the chain rule to take the time rate gives

$$\frac{d\sigma_{ij}}{dt} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \frac{d\varepsilon_{ij}}{dt} \tag{7}$$

or, in direct notation,

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}}, \text{ where } E_{ijkl} \equiv \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \tag{8}$$

The stiffness tensor $\underset{\approx\approx}{E}$ depends on the strain, but not on the strain *rate*. Therefore, nonlinear elasticity is always linear when expressed in rate form. We will later see that the set of equations governing nonlinear plasticity becomes linear in rate form, which permits us to solve for the rates. Thus, for numerical calculations, if the state is known at the beginning of a computational step, then knowing the rates permits us to predict the state at the end of the step.

Projection operations

Our goal is to demonstrate that equations of plasticity can be rearranged in a form that leads to a radial or oblique return algorithm. To interpret the solution geometrically, it is important to review the theory of orthogonal and oblique projection operators (also known as "idempotent" transformations). Ultimately, this theory will be applied in 6D stress space, but the geometric concepts will be introduced here using ordinary vectors in 3D space.

Orthogonal (high noon) projections. As sketched in Fig. 2, any vector $\underset{\sim}{x}$ can be projected onto a plane with unit normal $\underset{\sim}{n}$ by using the formula

$$\underset{\sim}{p} = \underset{\sim}{x} - \underset{\sim}{n}(\underset{\sim}{n} \bullet \underset{\sim}{x}) . \tag{9}$$

This operation is called an orthogonal projection because the projected vector $\underset{\sim}{p}$ represents the "shadow" cast by $\underset{\sim}{x}$ onto the plane perpendicular to $\underset{\sim}{n}$ when the light rays are coaxial with $\underset{\sim}{n}$. (Note: the term "coaxial" here means aligned, but not necessarily of the same directional sense.)

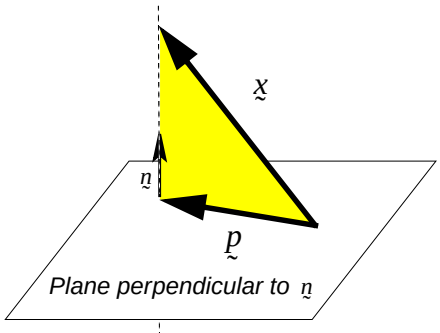

Figure 2. Orthogonal projection The path to the plane is the shortest distance.

Oblique (afternoon) projections. As sketched in Fig. 3, a more general kind of projection allows the "light rays" to intersect the plane at an *oblique* angle. This kind of projection can be characterized via two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Any vector $\underset{\sim}{b}$ that is perpendicular to the plane can be used to define the plane's orientation. The direction of the "light rays" can be characterized by any vector $\underset{\sim}{a}$ parallel to the light. Note that the magnitudes of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are inconsequential since only the orientations of those vectors carry relevant information. Of course, one can always demand that these vectors be unit vectors, but this is not necessary and often not convenient. As before, we seek an expression for the vector $\underset{\sim}{p}$ that is the projection of $\underset{\sim}{x}$ onto the plane. This time, however, we want the projection direction aligned with the vector $\underset{\sim}{a}$. Referring to Fig. 3, we can see that there must exist a scalar multiplier η such that the vector $\underset{\sim}{x}$ can be written

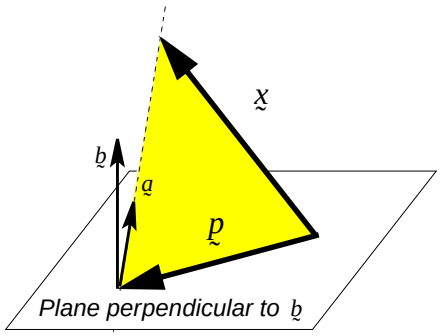

Figure 3. Oblique projection. The path obliquely intersects the plane.

$$\underset{\sim}{x} = \underset{\sim}{p} + \eta \underset{\sim}{a} , \tag{10}$$

To find the value of η, we impose the condition that the vector $\underset{\sim}{p}$ must lie in the plane. In other words, $\underset{\sim}{b} \bullet \underset{\sim}{p}$ must be zero to make $\underset{\sim}{p}$ perpendicular to $\underset{\sim}{b}$. Dotting both sides of Eq. (10) by $\underset{\sim}{b}$ (and setting $\underset{\sim}{b} \bullet \underset{\sim}{p} = 0$) gives

$$\underset{\sim}{b} \bullet \underset{\sim}{x} = 0 + \eta(\underset{\sim}{b} \bullet \underset{\sim}{a}). \tag{11}$$

Solving for η and substituting the result back into (10) gives the desired formula for the oblique projection:

$$\underset{\sim}{p} = P(\underset{\sim}{x}), \text{ where } \boxed{P(\underset{\sim}{x}) = \underset{\sim}{x} - \frac{\underset{\sim}{a}(\underset{\sim}{b} \bullet \underset{\sim}{x})}{\underset{\sim}{a} \bullet \underset{\sim}{b}}}. \tag{12}$$

Naturally, Eq. (9) is a special case of the more general Eq. (12), obtained by choosing $\underset{\sim}{a}=\underset{\sim}{b}=\underset{\sim}{n}$. In other words, the projection is orthogonal only if $\underset{\sim}{a}$ is proportional to $\underset{\sim}{b}$. The projector operator P in Eq. (12) is homogeneous of degree zero with respect to $\underset{\sim}{a}$ and/or $\underset{\sim}{b}$. (*i.e.*, multiplying either of these vectors by any nonzero scalar does not affect the formula). This property does *not* mean that the projection formula depends on only the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. The scaling property simply implies that the formula is independent of the magnitudes and senses of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.*

Figure (4) shows two vectors, $\underset{\sim}{x}$ and $\underset{\sim}{y}$, that fall on the line defined by $\underset{\sim}{a}$. More precisely, for some scalar β,

$$\underset{\sim}{x} = \underset{\sim}{y} + \beta\underset{\sim}{a}. \tag{13}$$

As seen in the sketch (or as verified by direct substitution into Eq. (12)),

$$P(\underset{\sim}{x})=P(\underset{\sim}{y}). \tag{14}$$

Conversely, if (14) holds, then so does (13).

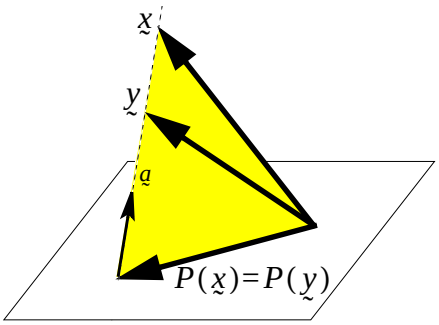

Figure 4. Projections of two vectors on the line defined by $\underset{\sim}{a}$.

Interpreted more geometrically, if two vectors have the same "shadow," then those they must differ from each other by some vector parallel to the "light rays."

* If desired, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ can be scaled such that $\underset{\sim}{a} \bullet \underset{\sim}{b}=1$. If this is done, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ would then be called "dual" vectors and the component form of Eq. (12) would take a particularly simple form when expressed using a nonorthogonal basis having $\underset{\sim}{g}_1 = \underset{\sim}{a}$ and $\underset{\sim}{g}^1 = \underset{\sim}{b}$. Namely $P(\underset{\sim}{x}) = x^2\underset{\sim}{g}_2 + x^3\underset{\sim}{g}_3$.

Analog of projections for tensors. For second-order tensors, the analog of Eq. (12) is the similarly-interpreted fourth-order oblique projection operation

$$P(\underset{\approx}{X}) = \underset{\approx}{X} - \frac{\underset{\approx}{A}(\underset{\approx}{B}:\underset{\approx}{X})}{\underset{\approx}{A}:\underset{\approx}{B}}. \tag{15}$$

As for the projection in 3-space, this operation represents a linear oblique projection in tensor space. The "surface" to which $\underset{\approx}{X}$ is projected is orthogonal to $\underset{\approx}{B}$ and the oblique projection direction is aligned with $\underset{\approx}{A}$. This projection function can be readily verified to have the following properties:

$$P(\alpha\underset{\approx}{X}) = \alpha P(\underset{\approx}{X}) \text{ for all scalars } \alpha. \tag{16}$$

$$P(\underset{\approx}{X} + \underset{\approx}{Y}) = P(\underset{\approx}{X}) + P(\underset{\approx}{Y}) \text{ for all } \underset{\approx}{X} \text{ and } \underset{\approx}{Y}. \tag{17}$$

$$P(P(\underset{\approx}{X})) = P(\underset{\approx}{X}). \tag{18}$$

The first two properties simply indicate that the projection operation is linear. The last property says that projecting a tensor that has already been projected merely gives the tensor back unchanged.

Finally, the analog of Eqs. (13) and (14) is the important identity that

$$P(\underset{\approx}{X}) = P(\underset{\approx}{Y}) \quad \text{if and only if} \quad \underset{\approx}{X} = \underset{\approx}{Y} + \beta\underset{\approx}{A}. \tag{19}$$

When we later encounter projection operations in the solution of plasticity equations, the tensor $\underset{\approx}{B}$ will be the gradient of the yield function. Consequently, $\underset{\approx}{B}$ will be normal to the yield surface in stress space. In the numerical plasticity solution, the operand $\underset{\approx}{X}$ in Eq. (15) will be the *trial elastic stress*, and the result $P(\underset{\approx}{X})$ will be the *actual updated stress*, projected back to the yield surface. We will find that the tensor $\underset{\approx}{A}$ will depend in a particular manner on the elastic stiffness and the plastic flow rule, and it cannot therefore be chosen at will. The tensor $\underset{\approx}{A}$ will not generally be parallel to the yield surface normal $\underset{\approx}{B}$. Consequently, we will find that the projection of the trial stress back to the yield surface is most properly accomplished by an *oblique* projection.

Governing equations for classical nonhardening plasticity

Classical nonhardening plasticity is usually presented in rate or incremental form so that the equations are linear in the rates. For typical numerical applications, the current stress $\underset{\sim}{\sigma}$ in a material and the strain rate $\dot{\underset{\sim}{\varepsilon}}$ (obtained by a symmetric gradient of the velocity field) are presumed known at the beginning of a time step. The rate at which the stress changes in response to the applied strain rate is desired. An interval of deformation is called "elastic" if the strain *would* return to its initial state (at the beginning of the interval) if the stress *were* to be released back to its value at the beginning of the interval. Otherwise, the interval of deformation is "plastic," and is associated with irreversible structural changes in the material. If the point at which a material transitions from elastic to plastic behavior depends only on the stress level, then the material is said to be nonhardening. The most simple (Von Mises) criterion states that plastic behavior commences when the magnitude of the stress deviator reaches a critical value. Other models (Tresca) might mark the onset of plasticity by reaching a critical maximum shear stress. Many geological and porous metal models further presume that the yield stress further depends on the amount of pressure as well as the stress deviator. Sometimes the onset of plasticity is delayed when the material is under sufficiently large confining pressure. If the material is anisotropic, then the onset of plasticity might require knowledge of the full stress tensor. The most general description of *nonhardening* plasticity presumes that there exists a scalar-valued "yield" function $f(\underset{\sim}{\sigma})$ such that negative values of f correspond to elastic stress states and positive values of f correspond to "forbidden" unattainable stress states. During intervals of plastic deformation, the yield function must be zero and remain zero. This requirement is called the "consistency" condition:

$$f(\underset{\sim}{\sigma})=0 \text{ and } \dot{f}=0 \text{ if the material is deforming plastically.} \tag{20}$$

The equation $f(\underset{\sim}{\sigma})=0$ defines a yield surface in 6-dimensional "stress space." By convention, the yield function f must be defined such that "elastic" stresses on the interior of this yield surface correspond to negative values of f and forbidden stresses on the exterior are identified by positive values of f. This convention is crucial because a trial stress is categorized to be elastic or plastic by checking the *sign* of the yield function. The upcoming analysis will be valid for arbitrarily anisotropic materials such as laminates or single crystals in which the *direction* of the stress is relevant. For plastically isotropic materials, the yield function depends only on the principal values of the stress, not on the eigenvectors. Many authors erroneously claim that the yield function for plastically isotropic materials can be alternatively written as a function of the standard stress invariants (thereby avoiding the need for an eigenvalue analysis). However, we have yet to see any such function that satisfies the require-

ment that $f < 0$ if and only if the stress state lies within the yield surface (*i.e.*, below yield). This issue is further addressed in Appendix 2, where the often-cited invariant form of the Tresca criterion is proved to be invalid (and therefore almost useless) for plasticity applications.

Elastic deformation. If the stress at a particular point in time is inside the yield surface (*i.e.*, if $f(\underset{\sim}{\sigma}) < 0$), then the deformation interval is elastic, and the stress rate is given by

$$\dot{\underset{\sim}{\sigma}} = \underset{\approx}{E} : \dot{\underset{\sim}{\varepsilon}}, \tag{21}$$

where $\underset{\approx}{E}$ is the elastic tangent stiffness tensor. If the material happens to be elastically isotropic, this equation becomes "Hooke's law" as described in Appendix 1. Equation (21) holds as long as the stress remains within the yield surface.

Plastic deformation. Now suppose the stress reaches the yield surface. If applying Eq. (21) would take the stress to forbidden states outside the yield surface, then the interval is plastic and the following plasticity equations (further explained in Appendix 2) are then imposed:

Consistency:	$\underset{\approx}{B} : \dot{\underset{\sim}{\sigma}} = 0$	(becomes Eq. 79 for hardening/softening)	(22)
Strain decomposition:	$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^p$		(23)
Flow rule:	$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda} \underset{\approx}{M}$		(24)
Nonlinear elasticity:	$\dot{\underset{\sim}{\sigma}} = \underset{\approx}{E} : \dot{\underset{\sim}{\varepsilon}}^e,$	(becomes Eq. 85 for elastic-plastic coupling)	(25)

In these equations, the following quantities are presumed known:

$\underset{\approx}{B}$, *gradient of the yield function at the current state* ($B_{ij} = \partial f / \partial \sigma_{ij}$)

$\dot{\underset{\sim}{\varepsilon}}$, *the total strain rate.*

$\underset{\approx}{E}$, *the fourth-order elastic tangent stiffness tensor.*

$\underset{\approx}{M}$, *the unit tensor in the direction of the plastic strain rate.*

The following quantities are unknown:

$\dot{\underset{\sim}{\sigma}}$, *the rate of stress*

$\dot{\underset{\sim}{\varepsilon}}^e$, *the elastic part of the strain rate*

$\dot{\underset{\sim}{\varepsilon}}^p$, *the plastic part of the strain rate.*

$\dot{\lambda}$, *the magnitude of the plastic part of the strain rate.*

We are interested in the *solution* of this set of equations, so we won't provide their rigorous derivations. We will, however, emphasize that Eq. (22) is just the rate form of the consistency condition $\dot{f} = 0$; specifically, if $f = f(\sigma_{ij})$, then the chain rule gives

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial \sigma_{ij}}\right)\frac{d\sigma_{ij}}{dt} = B_{ij}\dot{\sigma}_{ij} = 0, \tag{26}$$

which is the indicial form of Eq. (22). Since $\underset{\approx}{B}$ is the gradient of the yield function, it must be *normal* to the yield surface. Consequently, Eq. (22) states that the stress rate must be perpendicular to $\underset{\approx}{B}$. In other words, the stress rate must be tangent to the yield surface during plastic intervals. This condition makes geometrical sense because the stress state must remain on the yield surface during plastic intervals. The stress rate has no normal component because, for nonhardening plasticity, the yield surface itself has no normal velocity. Equation (23) expresses the typical notion that the total strain rate can be decomposed additively into elastic and plastic parts. The plastic strain rate tensor is not generally known at the outset, but Eq. (24) expresses the typical plasticity assumption that only the *magnitude* $\dot{\lambda}$ of the plastic strain rate is unknown — the *direction* of the plastic strain is presumed known (from a supplemental "flow" rule). For so-called "associative" plasticity, the plastic strain rate is assumed to be normal to the yield surface (in which case $\underset{\approx}{M}$ would be simply $\underset{\approx}{B}/\|\underset{\approx}{B}\|$). Finally, Eq. (25) reflects the assumption that the stress tensor is determined uniquely by the elastic strain tensor, and the stress *rate* therefore must be linear with respect to the elastic strain *rate*. This does *not* preclude nonlinear elastic response [recall Eq. (7)].

Range of validity of foregoing equations. Equations (22) through (25) are valid for the following conditions.

- Arbitrary elastic anisotropy.
- Arbitrary plastic anisotropy.
- Nonhardening yield surface. (This restriction is released later).
- Genuine nonlinear elasticity. In other words, the stress is *truly* a proper function of the elastic strain. The function may permissibly be nonlinear. Because the function is assumed proper, the stress *rate* will be linear in the strain *rate*, where the linear transformation is given by the elastic tangent stiffness tensor, which depends only on the elastic strain.
- Yield functions that obey the sign convention that elastic stresses correspond to negative values and forbidden stresses correspond to positive values (this is needed so that the outward normal can indeed be given by the yield function gradient and so that trial stresses may be categorized according to the sign of the yield function).
- Strain definitions that permit the decomposition of strain rates.
- Stress and strain definitions that permit the use of true rates rather than objective rates. A popular choice is to use the "unrotated" reference configuration.

- Strain rate *direction* being dependent only on the material state, not on the rate of change of state.

Notice that we do *not* require that the yield surface be convex or that the plastic strain rate be directed away from the yield surface. We don't even require that the elastic stiffness be positive definite. Rational models will indeed have such properties, but those concerns merely dictate appropriate choices for the quantities listed as "known" on page 12. To prove the radial and oblique return theorems, we will only need to presume that

$$\underset{\sim}{A} : \underset{\sim}{B} \neq \underset{\sim}{0}, \tag{27}$$

where

$$\underset{\sim}{A} \equiv \underset{\approx}{E} : \underset{\sim}{M} \tag{28}$$

No other assumptions about the "known" quantities will be needed.

Incidentally, for associated plasticity, the plastic flow direction $\underset{\sim}{M}$ is parallel to the yield surface normal $\underset{\sim}{B}$. Thus, for associativity, the condition of Eq. (27) becomes $\underset{\sim}{M} : \underset{\approx}{E} : \underset{\sim}{M} \neq \underset{\sim}{0}$, which is automatically satisfied if the elastic stiffness is positive definite.

For convenience, we will also assume that the elastic stiffness tensor is major symmetric:

$$E_{ijmn} = E_{mnij} \tag{29}$$

Exact solution of the classical nonhardening plasticity equations

In this section, the governing equations (22 through 25) for classical nonhardening rate-independent plasticity are rearranged so that they are seen to represent a projection of a trial elastic stress rate back to the yield surface. Geometrically, the tensor $\underset{\approx}{B}$ is proportional to the outward normal to the yield surface, and it will henceforth be referred to as the "normal" to the yield surface despite the fact that it might not have unit magnitude.

Eqs. (23) and (24) combine to give $\dot{\underset{\approx}{\varepsilon}}^e = \dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M}$, so that Eq. (25) becomes

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{E} : (\dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M}) . \tag{30}$$

For convenience, we write this in a more compact form,

$$\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \underset{\approx}{A}\dot{\lambda} , \tag{31}$$

where the so-called "trial" elastic stress rate $\dot{\underset{\approx}{\sigma}}^{\text{trial}}$ is defined

$$\dot{\underset{\approx}{\sigma}}^{\text{trial}} = \underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}} , \tag{32}$$

and $\underset{\approx}{A}$ is just a shorthand notation for

$$\underset{\approx}{A} \equiv \underset{\approx\approx}{E} : \underset{\approx}{M} . \tag{33}$$

Note that $\dot{\underset{\approx}{\sigma}}^{\text{trial}}$ and $\underset{\approx}{A}$ are both expressed in terms of *known* quantities, so they may themselves be regarded as known. All that remains is to use (22) to determine $\dot{\lambda}$. Substituting (31) into (22) gives

$$\underset{\approx}{B} : \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \dot{\lambda}(\underset{\approx}{B} : \underset{\approx}{A}) = 0 . \tag{34}$$

Thus

$$\boxed{\dot{\lambda} = \frac{\underset{\approx}{B} : \dot{\underset{\approx}{\sigma}}^{\text{trial}}}{\underset{\approx}{A} : \underset{\approx}{B}}} , \tag{35}$$

where we have used the fact that $\underset{\approx}{B} : \underset{\approx}{A} = \underset{\approx}{A} : \underset{\approx}{B}$. and we have invoked the assumption of Eq. (28). Substituting this result back into (31) gives the final solution for the stress rate in terms of known quantities:

$$\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \frac{\underset{\approx}{A}(\underset{\approx}{B} : \dot{\underset{\approx}{\sigma}}^{\text{trial}})}{\underset{\approx}{A} : \underset{\approx}{B}} . \tag{36}$$

To emphasize the structure of this solution, we write it compactly as

$$\dot{\underset{\sim}{\sigma}} = \begin{cases} P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}) & \text{during continued yielding} \\ \dot{\underset{\sim}{\sigma}}^{\text{trial}} & \text{during elastic intervals} \end{cases}, \tag{37}$$

where the operation P is defined

$$P(\underset{\sim}{X}) = \underset{\sim}{X} - \frac{\underset{\sim}{A}(\underset{\sim}{B}:\underset{\sim}{X})}{\underset{\sim}{A}:\underset{\sim}{B}} = P_{\underset{\sim}{A},\underset{\sim}{B}}(\underset{\sim}{X}). \tag{38}$$

The implicit dependence of the function P on $\underset{\sim}{A}$ and $\underset{\sim}{B}$ will be understood in the upcoming discussions. The key point is that the projection operator is constructed from *known* quantities, so it may itself be regarded as known.

Note from (33) and (24) that $\underset{\sim}{A}$ is proportional to $\underset{\approx}{E}:\dot{\underset{\sim}{\varepsilon}}^{p}$. Comparing Eq. (38) with Eq. (15) we recognize P as a projector! Hence, the exact solution has an appealing geometric interpretation:

> *During intervals of continued plastic deformation, the actual stress rate is obtained by obliquely projecting the trial stress rate to the yield surface. The projection direction is proportional to* $\underset{\approx}{E}:\dot{\underset{\sim}{\varepsilon}}^{p}$.

This result (which holds for any shape of yield surface) has been here derived under the assumption of nonhardening yield surfaces, but the analysis is generalized on page 26 to include hardening and softening. For hardening or softening yield surfaces, it will be seen that the actual stress rate is again a linear transformation of the trial stress rate, *but the transformation is no longer a projection*. Nevertheless, it will be demonstrated that the numerical return projection mapping algorithm is valid for *both* hardening and nonhardening yield functions so long as the target yield surface is updated *before* the projection is performed.

Detailed mathematical implications.* Note that the projection direction is proportional to $\underset{\approx}{E}:\dot{\underset{\sim}{\varepsilon}}^{p}$ and is *not* therefore generally proportional to the plastic strain rate $\dot{\underset{\sim}{\varepsilon}}^{p}$. A necessary and sufficient condition for the projection direction to be proportional to $\dot{\underset{\sim}{\varepsilon}}^{p}$ is $\underset{\approx}{E}:\dot{\underset{\sim}{\varepsilon}}^{p} = \alpha\dot{\underset{\sim}{\varepsilon}}^{p}$ for some scalar α. In other words, the projection direction will be proportional to the plastic strain rate *only if* the plastic strain rate is an eigentensor of the fourth-order stiffness tensor. For isotropic elasticity, this will be the case only if $\dot{\underset{\sim}{\varepsilon}}^{p}$ is purely deviatoric or purely isotropic (See Appendix 1). For many materials, plastic deformation causes no significant permanent volume changes, and therefore the plastic strain rate must be traceless (such an assumption is quite sensible for solid metals, but

*This subsection may be skipped without loss in continuity of later analyses.

grossly inaccurate for porous metals). Hence, isotropic elasticity in combination with plastic incompressibility forms a specialized sufficient condition for the return direction to be proportional to the plastic strain rate.

A large class of modern plasticity models takes the plastic strain rate to be normal to the yield surface (this assumption is often adopted for porous metals). Under the assumption of normality, the plastic strain rate $\dot{\underset{\approx}{\varepsilon}}^p$ is proportional to the yield surface normal $\underset{\approx}{B}$; in other words, $\dot{\underset{\approx}{\varepsilon}}^p = \alpha \underset{\approx}{B}$ for some scalar α. Consequently, the appropriate return direction is proportional to $\underset{\approx\approx}{E} : \underset{\approx}{B}$. Using similar reasoning as above, we conclude that the return direction will not generally be normal to the yield surface even if the plastic strain rate is normal to the yield surface; *i.e.*, the projection is generally oblique. A necessary and sufficient condition for normality of *both* the plastic strain rate and the return direction is $\underset{\approx\approx}{E} : \underset{\approx}{B} = \gamma \underset{\approx}{B}$ for some scalar γ. In other words, the normal to the yield surface must be an eigentensor of the fourth-order stiffness. In general, such a condition is not satisfied at all points on the yield surface. For isotropic elasticity, the only such points on the yield surface are located where the yield surface normal is either purely deviatoric or purely isotropic (see Appendix 1); this condition is satisfied everywhere for the non-pressure-dependent Von Mises yield surface discussed on page 24, but at *only at a subset* of points for the pressure-dependent Von Mises yield surface discussed on page 47.

To critically review an existing return algorithm, one can infer the *implicitly assumed plastic strain rate* by determining the projection direction used in the algorithm. We have shown that the projection direction $\underset{\approx}{A}$ is related to the plastic strain rate direction $\underset{\approx}{M}$ by $\underset{\approx}{A} = \underset{\approx\approx}{E} : \underset{\approx}{M}$. Thus, when examining an *existing* plastic return code for which you can tell what is used for the return direction $\underset{\approx}{A}$, you may conclude that the *implied* plastic strain rate direction is $\underset{\approx}{M} = \underset{\approx\approx}{F} : \underset{\approx}{A}$, where the compliance $\underset{\approx\approx}{F}$ is the inverse of the stiffness. For example, the so-called “radial” or “Prandtl” rule projects the stress to the yield surface by simply reducing the magnitude of the trial stress deviator. Thus, if the material is isotropic, the implied plastic strain rate is parallel to the stress deviator.

The projection direction is not generally aligned with the normal to the yield surface; *i.e.*, the projection is generally oblique. It’s natural to seek conditions for which the projection *will* be orthogonal (i.e., a projection to the nearest point on the yield surface). The projection will be orthogonal if and only if the projection direction $\underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}}^p$ is proportional to the yield surface normal $\underset{\approx}{B}$: *i.e.*, $\underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}}^p = \beta \underset{\approx}{B}$ for some scalar β. This conclusion is useful for interpreting

return algorithms that use orthogonal projections; these formulations correspond to an implied plastic strain rate that is proportional to $\underset{\approx}{\underset{\approx}{F}} : \underset{\approx}{B}$, where the compliance $\underset{\approx}{\underset{\approx}{F}}$ is the inverse of the stiffness.

Finally, note that the geometric interpretation of the exact solution does not rely directly on physical concerns such as plastic stability or maximum dissipation. These issues dictate appropriate choices for the quantities that have been presumed known ($\underset{\approx}{B}$, $\underset{\approx}{M}$, etc.) in the derivation of the solution. Even if bad choices are made, the exact solution of Eq. (37) remains unchanged.

Numerical solution by projection to the yield surface

For a numerical solution to the governing equations, the "known" quantities listed on page 12 are presumed known at the beginning of a time step. The governing equations are then solved numerically to update the state. All return algorithms work basically as follows: A trial stress is computed by taking the entire step to be elastic. If this trial stress happens to fall outside the yield surface (implying that there was actually some plastic deformation), then the assumption of elasticity was invalid and the actual updated stress is obtained by projecting the trial stress back to the yield surface. In this section, we rigorously justify this approach.

Recall the governing equation:

$$\dot{\underset{\sim}{\sigma}} = \begin{cases} P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}) & \text{during continued yielding} \quad \text{(a)} \\ \dot{\underset{\sim}{\sigma}}^{\text{trial}} & \text{during elastic intervals} \quad \text{(b)} \end{cases} . \tag{39}$$

In developing the numerical solution to this equation, it is imperative to keep in mind that the function P is a linear projector. During elastic intervals, the stress rate is identically equal to the trial elastic stress rate. During plastic intervals, the stress rate is the oblique projection of the trial elastic stress rate back to the yield surface. The projection direction is proportional to $\underset{\approx}{\underset{\sim}{E}} : \dot{\underset{\sim}{\varepsilon}}^{p}$.

This section details an efficient numerical algorithm for solution of the above equation. Following conventional finite difference protocol, the "known" quantities ($\underset{\sim}{B}$, $\dot{\underset{\sim}{\varepsilon}}$, $\underset{\approx}{\underset{\sim}{E}}$, and $\underset{\sim}{M}$, and therefore the projection operator P) are approximated as unchanging throughout the time step. The order of accuracy of the algorithm is affected by what values are assigned for these tensors. Typically, middle-of-step estimates improve the order of accuracy and help reduce numerical errors associated with moving the stress past high-curvature parts of the yield surface (*e.g.*, yield surface vertices).

Given an old stress state $\underset{\sim}{\sigma}^{\text{old}}$ and a time step Δt, the finite difference solution for the new stress state is

$$\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{old}} + \dot{\underset{\sim}{\sigma}} \Delta t . \tag{40}$$

The trial stress rate $\dot{\underset{\sim}{\sigma}}^{\text{trial}}$ is computed by applying Eq. (32). The trial *stress* $\underset{\sim}{\sigma}^{\text{trial}}$ is *constructed* to be the stress state achieved by presuming that the entire step is elastic. Thus, applying Eq. (39b), under the assumption of an elastic interval

$$\underset{\sim}{\sigma}^{\text{trial}} = \underset{\sim}{\sigma}^{\text{old}} + \dot{\underset{\sim}{\sigma}}^{\text{trial}} \Delta t . \tag{41}$$

As long as $\underset{\sim}{\sigma}^{\text{trial}}$ is an admissible stress (*i.e.*, if $f(\underset{\sim}{\sigma}^{\text{trial}}) \le 0$), then the assumption of elasticity was valid and $\underset{\sim}{\sigma}^{\text{new}}=\underset{\sim}{\sigma}^{\text{trial}}$, making the computational cycle complete. If, on the other hand, $\underset{\sim}{\sigma}^{\text{trial}}$ happens to fall in the forbidden zone outside the yield surface, then there must have been plastic deformation, and during at least part of the step, Eq. (39a) applies:

$$\dot{\underset{\sim}{\sigma}} = P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}) . \tag{42}$$

Whenever $\underset{\sim}{\sigma}^{\text{trial}}$ is found to equal a forbidden stress outside the yield surface, then we know that it does *not* represent the updated stress. Fortunately, however, the effort that went into computing the trial stress (which was considerable if the elasticity was anisotropic) need not go to waste. *By construction*, we know that $\underset{\sim}{\sigma}^{\text{trial}}$ is nevertheless still related to $\dot{\underset{\sim}{\sigma}}^{\text{trial}}$ by Eq. (41) and hence

$$\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t = \underset{\sim}{\sigma}^{\text{trial}} - \underset{\sim}{\sigma}^{\text{old}} . \tag{43}$$

During this time step, the stress state might have been initially below yield, and a portion Δt^e of the total time step may have been used to elastically get the stress to the yield surface. Breaking the step into elastic and plastic intervals, the stress at the end of the elastic interval is

$$\underset{\sim}{\sigma}^{\text{intermediate}} = \underset{\sim}{\sigma}^{\text{old}} + \dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^e . \tag{44}$$

Using the projected stress rate during the remaining plastic part of the time step, $\Delta t^p = \Delta t - \Delta t^e$, the final stress is

$$\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{intermediate}} + P(\dot{\underset{\sim}{\sigma}}^{\text{trial}})\Delta t^p , \tag{45}$$

or

$$\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{old}} + \dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^e + P(\dot{\underset{\sim}{\sigma}}^{\text{trial}})\Delta t^p . \tag{46}$$

Substituting $\Delta t^e = \Delta t - \Delta t^p$ in the second term and applying Eq. (16) to the last term gives

$$\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{old}} + \dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t - \dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p + P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p) \tag{47}$$

Using Eq. (43) to re-introduce the already available trial stress gives

$$\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{trial}} - \dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p + P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p) . \tag{48}$$

A fascinating advantage of the projector nature of the governing equations is that it is not necessary to find the exact time at which the deformation transitions from elastic to plastic! Specifically, operating on both sides of (48) with the projection function P and using linearity gives

$$P(\underset{\sim}{\sigma}^{\text{new}}) = P(\underset{\sim}{\sigma}^{\text{trial}}) - P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p) + P(P(\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p)) . \tag{49}$$

Applying Eq. (18), the last two terms cancel with each other, leaving only

$$P(\underset{\sim}{\sigma}^{\text{new}}) = P(\underset{\sim}{\sigma}^{\text{trial}}). \tag{50}$$

Recalling Eq. (19), this condition is possible if and only if $\underset{\sim}{\sigma}^{\text{new}}$ and $\underset{\sim}{\sigma}^{\text{trial}}$ fall on a line that is proportional to the projection direction $\underset{\sim}{A}$. As sketched in Fig. 5, once the trial stress is known, all that needs to be done is to project it back to the yield surface! This is true no matter what fraction of the time step is elastic. The projection direction is *required* to be proportional to $\underset{\sim}{A}$. Therefore, applying Eq. (19) to Eq. (42), we know that

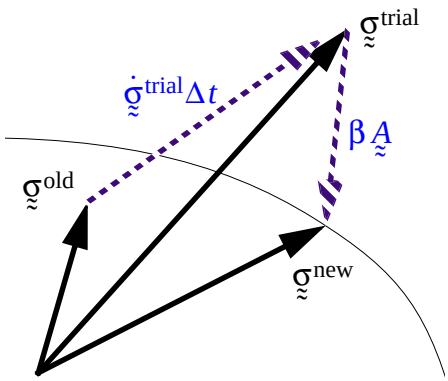

Figure 5. Return to the yield surface.

$$\boxed{\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{trial}} + \beta\,\underset{\sim}{A}}, \tag{51}$$

for some scalar β. Note that *the new updated stress is not generally a scalar multiple of the trial stress.* That would be the case only if the projection direction $\underset{\sim}{A}$ is in line with the trial stress.

Given that $\underset{\sim}{\sigma}^{\text{new}}$ is the final stress after an interval of plastic deformation, $\underset{\sim}{\sigma}^{\text{new}}$ must be on the yield surface. Hence, the scalar β is found by using, say, a secant solver to find β in the yield equation,

$$\boxed{f(\underset{\sim}{\sigma}^{\text{trial}} + \beta\,\underset{\sim}{A}) = 0}. \tag{52}$$

Recalling the explicit expression of the projector, the solution of Eq. (48) is

$$\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{trial}} - \frac{\underset{\sim}{A}(\underset{\sim}{B}:\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p)}{\underset{\sim}{A}:\underset{\sim}{B}} \tag{53}$$

Comparing with Eq. (51), a good first guess for the secant solver is

$$\beta^{(1)} = -\frac{(\underset{\sim}{B}:\dot{\underset{\sim}{\sigma}}^{\text{trial}}\Delta t^p)}{\underset{\sim}{A}:\underset{\sim}{B}} \qquad \text{<--- first guess} \tag{54}$$

Of course, the value of Δt^p is generally unknown, but the mean value theorem can provide a sensible estimate (for well-behaved yield functions):

$$\Delta t^p = \frac{\Delta t}{1 - f(\underset{\sim}{\sigma}^{\text{old}})/f(\underset{\sim}{\sigma}^{\text{trial}})} \qquad \text{<--- first guess} \tag{55}$$

Once the first-order solution is known, it might be a good practice to re-evaluate the "known" quantities $\underset{\sim}{A}$ and $\underset{\sim}{B}$ at the improved estimates for the stress:

$$\underset{\sim}{\sigma}^{\text{new}(1)} = \underset{\sim}{\sigma}^{\text{trial}} + \beta^{(1)}\underset{\sim}{A}^{(0)} \qquad \Rightarrow \underset{\sim}{A}^{(1)} \text{ and } \underset{\sim}{B}^{(1)} \tag{56}$$

Taking $\underset{\sim}{\sigma}^{(0)} \equiv \underset{\sim}{\sigma}^{\text{old}}$ and $\underset{\sim}{\sigma}^{(1)} = \underset{\sim}{\sigma}^{\text{trial}}$, the following iterative algorithm might be a higher-order alternative to a secant solution because it accounts for variation of the "known" $\underset{\sim}{A}$ and $\underset{\sim}{B}$ quantities with stress:

(i) Compute $\underset{\sim}{\sigma}^{\text{trial}}$.

(ii) If $f(\underset{\sim}{\sigma}^{\text{trial}}) \le 0$ then $\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{\text{trial}}$ and go to (ix).

(iii) Set $\underset{\sim}{\sigma}^{(0)} \equiv \underset{\sim}{\sigma}^{\text{old}}$

set $\underset{\sim}{\sigma}^{(1)} = \underset{\sim}{\sigma}^{\text{trial}}$

set $k = 0$

(iv) Increment counter k by 1.

(v) Estimate what fraction of the vector from $\underset{\sim}{\sigma}^{(k-1)}$ to $\underset{\sim}{\sigma}^{(k)}$ lies outside the yield surface (negative values are allowed for iteration purposes).

$$\eta = \frac{1}{[1 - f(\underset{\sim}{\sigma}^{(k-1)})/f(\underset{\sim}{\sigma}^{(k)})]}$$

(vi) Compute $\underset{\sim}{A}^{(k)}$ and $\underset{\sim}{B}^{(k)}$ using $\eta\underset{\sim}{\sigma}^{(k-1)} + (1-\eta)\underset{\sim}{\sigma}^{(k)}$ as a stress estimate. (This estimate is generally off the yield surface, so this algorithm requires a well-behaved yield function.)

(vii) Update stress estimate:

$$\underset{\sim}{\sigma}^{(k+1)} = \underset{\sim}{\sigma}^{(k)} - \eta\left[\frac{\underset{\sim}{A}^{(k)}(\underset{\sim}{B}^{(k)} : (\underset{\sim}{\sigma}^{(k)} - \underset{\sim}{\sigma}^{(k-1)}))}{\underset{\sim}{A}^{(k)} : \underset{\sim}{B}^{(k)}}\right]$$

(viii) If $f(\underset{\sim}{\sigma}^{(k+1)})=0$ then $\underset{\sim}{\sigma}^{\text{new}} = \underset{\sim}{\sigma}^{(k+1)}$ and go to step (ix). Otherwise, go to step (iv).

(ix) Stop.

An iterative solver like this may not be necessary if Eq. (52) is simple enough to solve for β analytically. The classical Von Mises yield criterion discussed on page 24 is such a case. For pressure dependent yield surfaces, a far more detailed alternative algorithm is given in Appendix 3 on page 5.

The solution of Eqs. (51) and (52) is *exact* if the "known" quantities listed on page 12 are truly constant over the entire interval. The algorithm should be highly accurate for typical yield surfaces and flow rules so long as the tensors $\underset{\sim}{A}$ and $\underset{\sim}{B}$ don't vary too much over the interval. If, for example, the stress state is on a "flat" part of the yield surface, then the direction of $\underset{\sim}{B}$ is exactly

constant. A continuing area of research addresses what to do near regions of high curvature in the yield surface where $\underset{\approx}{A}$ and $\underset{\approx}{B}$ cannot sensibly be assumed constant.

Traditional Von Mises radial return

The Von Mises yield condition says that yield occurs when the second invariant of the stress deviator reaches a critical positive constant k^2. The corresponding yield function is

$$f(\underset{\approx}{\sigma}) = \frac{1}{2}(\underset{\approx}{S}:\underset{\approx}{S}) - k^2, \text{ where } \underset{\approx}{S} \equiv \underset{\approx}{\sigma} - \tfrac{1}{3}\text{trace}(\underset{\approx}{\sigma})\underset{\approx}{\boldsymbol{I}}. \tag{57}$$

For traditional Von Mises plasticity, k is a material constant independent of the stress. Using elementary tensor calculus, the gradient of this yield function is found to be $\underset{\approx}{B}=\underset{\approx}{S}$. Hence, the outward *unit* normal to the yield surface at a stress state $\underset{\approx}{\sigma}$ on the yield surface is $\underset{\approx}{S}$ divided by its own magnitude:

$$\underset{\approx}{N} = \frac{\underset{\approx}{B}}{\sqrt{\underset{\approx}{B}:\underset{\approx}{B}}} = \frac{\underset{\approx}{S}}{\sqrt{\underset{\approx}{S}:\underset{\approx}{S}}} = \frac{\underset{\approx}{S}}{k\sqrt{2}}. \tag{58}$$

Geometrically, the Von Mises yield surface is a cylinder in stress space, where the "axis" is the set of isotropic tensors, and the stress deviator is oriented in a purely radial direction.

Traditional Von Mises plasticity takes the flow rule to be associative, and the direction of the plastic strain rate is therefore aligned with $\underset{\approx}{N}$.

$$\underset{\approx}{M} = \underset{\approx}{N}. \tag{59}$$

Note from Eq. (58) that $\underset{\approx}{N}$ and therefore $\underset{\approx}{M}$ are deviatoric. Consequently, for isotropic elasticity, $\underset{\approx}{M}$ is an eigentensor of the stiffness (See Appendix 1). In other words, the projection direction $\underset{\approx}{A}$ and the plastic strain rate happen to be collinear for this simple model. Namely, using Eq. (120) in Appendix 1,

$$\underset{\approx}{A} = \underset{\approx\approx}{E}:\underset{\approx}{M} = 2G\underset{\approx}{M}. \tag{60}$$

Hence, classical Von Mises plasticity is an "exceptional" model having both associativity of the plastic strain rate and a proportional orthogonal projection back to the yield surface. Recall that $\underset{\approx}{M}$ is proportional to $\underset{\approx}{N}$, which is in turn proportional to $\underset{\approx}{S}$. Therefore projection direction $\underset{\approx}{A}$ is proportional to the stress deviator $\underset{\approx}{S}$. Typically, Von Mises radial return methods take the tensor $\underset{\approx}{A}$ to equal its value at the end of the time step so that Eq. (51) becomes

$$\underset{\approx}{\sigma}^{\text{new}} = \underset{\approx}{\sigma}^{\text{trial}} + \beta\underset{\approx}{S}^{\text{new}}, \tag{61}$$

where the scalar β must be assigned to put the new stress on the yield surface. Taking the deviatoric part of both sides of (61) shows that

$$\underset{\approx}{S}^{\text{new}} = \underset{\approx}{S}^{\text{trial}} + \beta\underset{\approx}{S}^{\text{new}}, \tag{62}$$

or, redefining the undetermined constant,

$$\underset{\approx}{S}^{\text{new}} = \psi \underset{\approx}{S}^{\text{trial}}, \tag{63}$$

for some scalar ψ. In other words, the new stress deviator is simply a scalar multiple of the trial stress deviator. This result is particularly nice because it allows Eq. (52) to be solved analytically. Once the trial stress is known, the scalar multiple ψ must be constructed to put the stress on the yield surface.

Therefore using Eq. (63) in (57) and setting $f(\underset{\approx}{\sigma}^{\text{new}}) = 0$ gives

$$\boxed{\underset{\approx}{S}^{\text{new}} = \psi \underset{\approx}{S}^{\text{trial}} \text{ where } \psi = \sqrt{\frac{2k^2}{\underset{\approx}{S}^{\text{trial}} : \underset{\approx}{S}^{\text{trial}}}}.} \tag{64}$$

Taking the isotropic part of both sides of (61) shows that the new updated pressure simply equals the pressure associated with the trial stress:

$$\boxed{p^{\text{new}} = p^{\text{trial}}}. \tag{65}$$

If desired, the plastic strain rate may be computed using STEP 21 in the algorithm on page 59.

Hardening/Softening yield surfaces

For hardening (or softening) plasticity, Eqs. (23) through (25) remain valid:

$$\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^e + \dot{\underset{\sim}{\varepsilon}}^p \tag{66}$$

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda}\underset{\sim}{M} \tag{67}$$

$$\dot{\underset{\sim}{\sigma}} = \underset{\approx}{E} : \dot{\underset{\sim}{\varepsilon}}^e, \tag{68}$$

However, Eq. (22) must be modified to account for the fact that the yield surface is no longer strictly a function of only the stress. The yield function is additionally dependent on one or more internal state variables. For example, plastic deformation in metals can cause structural changes such as dislocation buildups that change the Von Mises yield stress k in Eq. (57); and therefore, k would be interpreted as an internal state variable. Sufficiently large compression of porous materials causes a mesoscopic structural change (pore crushup) and the yield surface would naturally depend on porosity as an internal state variable. For illustration purposes, we will show the analysis for the case of a set of internal state variables, $\{\eta_1, \eta_2, \ldots\}$. Thus, the equation defining the yield surface becomes

$$f(\underset{\sim}{\sigma}, \eta_1, \eta_2, \ldots) = 0. \tag{69}$$

Exactly one of the following always holds:

$$\begin{aligned} &\textit{Elastic deformation: } \dot{\lambda} = 0 \text{ and } f(\underset{\sim}{\sigma}, \eta_1, \eta_2, \ldots) \le 0 \\ &\textit{Plastic deformation: } \dot{\lambda} > 0 \text{ and } f(\underset{\sim}{\sigma}, \eta_1, \eta_2, \ldots) = 0 \text{ and } \dot{f} = 0 \end{aligned} \tag{70}$$

These conditions are described succinctly by the "Kuhn-Tucker" complementary conditions, which must hold regardless of whether the deformation is elastic or plastic:

$$\begin{aligned} &\dot{\lambda} \ge 0, \\ &f(\underset{\sim}{\sigma}, \eta_1, \eta_2, \ldots) \le 0, \\ &\dot{\lambda} f(\underset{\sim}{\sigma}, \eta_1, \eta_2, \ldots) = 0, \text{ and} \\ &\dot{\lambda}\dot{f} = 0 \end{aligned} \tag{71}$$

Taking the rate of Eq. (69), the consistency condition for continued yield ($\dot{f} = 0$) becomes

$$\underset{\sim}{B} : \dot{\underset{\sim}{\sigma}} + \beta_1\dot{\eta}_1 + \beta_2\dot{\eta}_2 + \ldots = 0. \tag{72}$$

where, as before, $\underset{\sim}{B}$ represents the gradient of the yield function in stress space,

$$B_{ij} = \frac{\partial f}{\partial \sigma_{ij}}, \tag{73}$$

The new β_k quantities for hardening/softening quantify how the yield surface changes upon a change in the internal state variables:

$$\beta_1 \equiv \frac{\partial f}{\partial \eta_1}, \qquad \beta_2 \equiv \frac{\partial f}{\partial \eta_2}, \qquad \text{etc.} \tag{74}$$

Eq. (72) shows that the yield surface is no longer fixed in stress space. Changes in the internal state variables cause the yield surface to move, and the stress rate during plastic intervals is therefore no longer required to be tangent to the yield surface. In order for the stress to remain on the moving yield surface, the stress rate must have a normal component that exactly equals the expansion (or contraction) speed of the yield surface. This condition is ensured by imposing the consistency condition of Eq. (72).

Whenever internal state variables are introduced, additional equations known as "evolution" equations must be supplied that govern the rate of change of the internal state variables.

Example of an evolution equation. In this example, we describe a particular example of an internal state variable and show how its evolution equation can be expressed in the form of a second-order tensor operating on the *plastic part* of the strain rate. Consider porous metals, which exhibit significant permanent volume changes even when the matrix material is plastically incompressible. In an unstressed representative sample of volume V, the volume of the matrix material is $V - \phi V$, where ϕ is the porosity at the unstressed state. During an interval of elastic deformation the volume of the pores and the volume of the matrix material can both change, not necessarily in proportion to each other. Nevertheless, the pore and matrix volumes *in the unstressed reference state* remain constant during elastic deformation. Hence, the *unstressed reference* porosity is a natural choice as an internal state variable because it never changes during purely elastic deformation. By using the unstressed value, our internal state variable can change only due to plastic deformation. We seek an evolution equation that governs the rate $\dot{\phi}$ of this unstressed elastic reference porosity. If the matrix material is plastically incompressible, any changes in the macroscopic unloaded volume must be attributable to changes in the elastic unloaded porosity. In other words, the unstressed reference volume of the *matrix* material, $V - \phi V$, must be constant. Thus $\dot{V} - \dot{\phi} V - \phi \dot{V} = 0$. Dividing both sides by V and using the continuum mechanics identity that $\dot{V}/V = \mathrm{tr}\dot{\underset{\approx}{\varepsilon}}^p$, we obtain the desired evolution equation for the porosity state variable. Namely,

$$\dot{\phi} = (1-\phi)\mathrm{tr}\dot{\underset{\approx}{\varepsilon}}^p, \qquad \text{or } \dot{\phi} = \hat{\underset{\sim}{G}} : \dot{\underset{\approx}{\varepsilon}}^p \text{ where } \hat{\underset{\sim}{G}} = (1-\phi)\underset{\approx}{I} \tag{75}$$

Note that the rate of the internal state variable is expressed as a "known" state-dependent tensor, $\hat{\underset{\sim}{G}}$, operating on the plastic strain rate.

An equivalent form for evolution equations. Another typical approach for expressing the evolution relation for an internal state variable η_k is to assume

$$\dot{\eta}_k = -\dot{\lambda} m_k , \tag{76}$$

where m_k is some specified state-dependent scalar and $\dot{\lambda}$ is the plasticity parameter of Eq. (24). This is essentially the form cited by Simo and Hughes [><] in their *Eq. (2.2.8)* where they use the symbols $\boldsymbol{q}$, γ, and $\boldsymbol{h}$ in place of our η_k, $\dot{\lambda}$, and m_k, respectively. Recalling that the tensor $\underset{\sim}{M}$ in Eq. (24) is a unit tensor, it can be double-dotted into both sides of (24) to give $\dot{\lambda} = \underset{\sim}{M} : \dot{\underset{\sim}{\varepsilon}}^p$, and therefore Eq. (76) can be written

$$\dot{\eta}_k = \hat{\underset{\sim}{G}}_k : \dot{\underset{\sim}{\varepsilon}}^p , \text{ where } \hat{\underset{\sim}{G}}_k = -m_k \underset{\sim}{M} . \tag{77}$$

Though not immediately obvious, the example of Eq. (75) is a special case of Eq. (76) with $m = -(1-\phi)\mathrm{tr}\underset{\sim}{M}$.

Application to hardening/softening. Note that the fundamental structure of Eqs. (75) and (77) is identical. Namely, the rate of the internal state variable equals the inner product of a state dependent tensor $\hat{\underset{\sim}{G}}$ into the plastic strain rate $\dot{\underset{\sim}{\varepsilon}}^p$. Such a form for the evolution equations is extremely common. In the subsequent analysis, we therefore assume that, for each internal state variable η_k, there exists a state-dependent second-order tensor $\hat{\underset{\sim}{G}}_k$ such that the evolution equation for the state variable may be written

$$\dot{\eta}_k = \hat{\underset{\sim}{G}}_k : \dot{\underset{\sim}{\varepsilon}}^p . \tag{78}$$

Thankfully, it will be shown that it is not necessary to *actually compute* the $\hat{\underset{\sim}{G}}_k$ tensors when performing numerical solutions. To prove the algorithm, we merely need to assert that *they exist*.

Given Eq. (78), we can imagine the existence of a state-dependent "ensemble" evolution tensor $\underset{\sim}{G} \equiv \beta_1 \hat{\underset{\sim}{G}}_1 + \beta_2 \hat{\underset{\sim}{G}}_2 + \ldots$ such that the consistency Eq. (72) becomes

$$\boxed{\underset{\sim}{B} : \dot{\underset{\sim}{\sigma}} + \underset{\sim}{G} : \dot{\underset{\sim}{\varepsilon}}^p = 0} . \tag{79}$$

Equation (79) is the generalization of Eq. (22).

Elastic-plastic coupling.

A simple way to explain elastic plastic coupling is to first presume that the elastic strain tensor $\underset{\sim}{\varepsilon}^e$ is a function of one or more of the η_k internal state variables in addition to being a function of stress:

$$\dot{\underset{\approx}{\varepsilon}}^e = \dot{\underset{\approx}{\varepsilon}}^e(\underset{\approx}{\sigma}, \eta_1, \ldots, \eta_k) \tag{80}$$

The rate form of Eq. (80) is

$$\dot{\underset{\approx}{\varepsilon}}^e = \underset{\approx\approx}{\boldsymbol{F}} : \dot{\underset{\approx}{\sigma}} + \sum_k \mu_k \dot{\eta}_k \tag{81}$$

where we have defined the elastic compliance tensor to be

$$\underset{\approx\approx}{\boldsymbol{F}} \equiv \left(\frac{\partial \underset{\approx}{\varepsilon}^e}{\partial \underset{\approx}{\sigma}} \right)_{\eta} \tag{82}$$

and

$$\underset{\approx}{\mu}_k \equiv \left(\frac{\partial \underset{\approx}{\varepsilon}^e}{\partial \eta_k} \right)_{\underset{\approx}{\sigma}} \tag{83}$$

The elastic compliance $\underset{\approx\approx}{\boldsymbol{F}}$ and the elastic stiffness,

$$\underset{\approx\approx}{\boldsymbol{E}} \equiv \left(\frac{\partial \underset{\approx}{\sigma}}{\partial \underset{\approx}{\varepsilon}^e} \right)_{\eta} = \underset{\approx\approx}{\boldsymbol{F}}^{-1}, \tag{84}$$

now depend on the values of the internal state variables. Multiplying both sides of (81) by the stiffness and then solving for the stress rate gives

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{\boldsymbol{E}} : \dot{\underset{\approx}{\varepsilon}}^e - \sum_k \left(\underset{\approx\approx}{\boldsymbol{E}} : \underset{\approx}{\mu}_k \right) \dot{\eta}_k \tag{85}$$

Now let's recall that the plastic internal state variables are assumed to be expressible in the form

$$\dot{\eta}_k = \hat{\underset{\approx}{G}}_k : \dot{\underset{\approx}{\varepsilon}}^p . \tag{86}$$

so that

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{\boldsymbol{E}} : \dot{\underset{\approx}{\varepsilon}}^e - \sum_k \left(\underset{\approx\approx}{\boldsymbol{E}} : \underset{\approx}{\mu}_k \right) \hat{\underset{\approx}{G}}_k : \dot{\underset{\approx}{\varepsilon}}^p \tag{87}$$

or, writing $\dot{\underset{\approx}{\varepsilon}}^p = \dot{\lambda} \underset{\approx}{M}$,

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{\boldsymbol{E}} : \dot{\underset{\approx}{\varepsilon}}^e - \underset{\approx}{\boldsymbol{z}} \dot{\lambda} \tag{88}$$

where

$$\underset{\approx}{\boldsymbol{z}} \equiv \sum_k \left(\underset{\approx\approx}{\boldsymbol{E}} : \underset{\approx}{\mu}_k \right) \hat{\underset{\approx}{G}}_k : \underset{\approx}{M} \tag{89}$$

Again, we want to emphasize that this ugly expression never has to be actually computed in practice. The key point here is that there exists a second-order tensor $\underset{\approx}{\boldsymbol{z}}$ that depends only on the material state (not its rate) such that Eq. (88) holds.

This is the generalization of Eq. (25) when there is elastic-plastic coupling.

Physical example: porosity. Let's consider a model with only one internal state variable — the porosity $\eta = \phi$. A very common form for the porosity-dependence in the elastic compliance is

$$\underset{\approx}{\boldsymbol{F}} = \underset{\approx}{\boldsymbol{F}}^{\text{solid}} + \frac{\phi}{1-\phi}\underset{\approx}{\boldsymbol{F}}^{\text{c}} \tag{90}$$

Here $\underset{\approx}{\boldsymbol{F}}^{\text{solid}}$ is the compliance of the solid (nonporous) matrix material, and $\underset{\approx}{\boldsymbol{F}}^{\text{c}}$ is a coupling tensor that is independent of the porosity. The linear elastic stress-strain relation is

$$\underset{\approx}{\varepsilon}^{e} = \underset{\approx}{\boldsymbol{F}} : \underset{\approx}{\sigma} = \left[\underset{\approx}{\boldsymbol{F}}^{\text{solid}} + \psi\underset{\approx}{\boldsymbol{F}}^{\text{c}}\right] : \underset{\approx}{\sigma} \tag{91}$$

from which it follows that

$$\underset{\approx}{\mu} \equiv \left(\frac{\partial \underset{\approx}{\varepsilon}^{e}}{\partial \eta}\right)_{\sigma} = \left(\frac{\partial \underset{\approx}{\varepsilon}^{e}}{\partial \phi}\right)_{\sigma} = \frac{1}{(1-\phi)^2}\left(\underset{\approx}{\boldsymbol{F}}^{\text{c}} : \underset{\approx}{\sigma}\right) \tag{92}$$

Incidentally, recognizing that the elastic strain tensor is not itself usually available in numerical implementations, we presented this example of coupling by using compliances instead of stiffnesses so that the final expression for $\underset{\approx}{\mu}$ would depend only on the stress. The stress is readily available in numerical calculations because it is always needed to solve the momentum equation. Thus Eq. (92) is quite convenient because it permits us to avoid having to additionally compute and allocate storage for the strain tensor.

The z-tensor is computed by substituting Eq. (92) into (89):

$$\underset{\approx}{\boldsymbol{z}} \equiv \frac{1}{(1-\phi)^2}\left(\underset{\approx}{\boldsymbol{E}} : \underset{\approx}{\boldsymbol{F}}^{\text{c}} : \underset{\approx}{\sigma}\right)\hat{\underset{\approx}{G}}_k : \underset{\approx}{M} \tag{93}$$

or, recalling Eq. (75),

$$\underset{\approx}{\boldsymbol{z}} \equiv \frac{1}{(1-\phi)}\left(\underset{\approx}{\boldsymbol{E}} : \underset{\approx}{\boldsymbol{E}}^{\text{c}} : \underset{\approx}{\sigma}\right)\text{tr}\,\underset{\approx}{M} \quad . \tag{94}$$

Again, we want to emphasize that we have derived the z-tensor for illustration purposes only. Fortunately, there is really no need to actually compute it in practice. The purpose of introducing the z-tensor was to demonstrate that such a tensor *exists* such that the structural form of Eq. (88) holds. Only the existence of the z-tensor is needed in order to demonstrate that the radial return algorithm applies to situations of elastic-plastic coupling. We will now proceed to demonstrate how such coupling affects the *direction* that one must return to the yield surface.

The generalized governing equations. The other governing equations, namely (23) and (24), remain unchanged.

Consistency:	$\underset{\approx}{B}:\dot{\underset{\approx}{\sigma}} + \underset{\approx}{G}:\dot{\underset{\approx}{\varepsilon}}^p = 0$	(95)
Strain decomposition:	$\dot{\underset{\approx}{\varepsilon}} = \dot{\underset{\approx}{\varepsilon}}^e + \dot{\underset{\approx}{\varepsilon}}^p$	(96)
Flow rule:	$\dot{\underset{\approx}{\varepsilon}}^p = \dot{\lambda}\underset{\approx}{M}$	(97)
Nonlinear elasticity:	$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{\boldsymbol{E}}:\dot{\underset{\approx}{\varepsilon}}^e - \underset{\approx}{z}\dot{\lambda}$	(98)

Eqs. (96) and (97) combine to give $\dot{\underset{\approx}{\varepsilon}}^e = \dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M}$, so that Eq. (98) becomes

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{\boldsymbol{E}}:(\dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M}) - \underset{\approx}{z}\dot{\lambda}. \tag{99}$$

For convenience, we write this in a more compact form,

$$\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \underset{\approx}{A}\dot{\lambda}, \tag{100}$$

where the so-called "trial" elastic stress rate $\dot{\underset{\approx}{\sigma}}^{\text{trial}}$ is again defined

$$\dot{\underset{\approx}{\sigma}}^{\text{trial}} = \underset{\approx\approx}{\boldsymbol{E}}:\dot{\underset{\approx}{\varepsilon}}, \tag{101}$$

and $\underset{\approx}{A}$ is a shorthand notation for

$$\underset{\approx}{A} \equiv \underset{\approx\approx}{\boldsymbol{E}}:\underset{\approx}{M} - \underset{\approx}{z}\dot{\lambda}. \tag{102}$$

Note that the $\underset{\approx}{A}$ tensor is now defined with a new term that accounts for elastic-plastic coupling. As for the simpler nonhardening non-coupled case, note that $\dot{\underset{\approx}{\sigma}}^{\text{trial}}$ and $\underset{\approx}{A}$ are both expressed in terms of *known* quantities, so they may themselves be regarded as known. All that remains is to use (22) to determine $\dot{\lambda}$. Substituting (100) into (95) gives

$$\underset{\approx}{B}:(\dot{\underset{\approx}{\sigma}}^{\text{trial}} - \underset{\approx}{A}\dot{\lambda}) + \underset{\approx}{G}:\dot{\underset{\approx}{\varepsilon}}^p = 0. \tag{103}$$

Thus solving for the plasticity parameter $\dot{\lambda}$ gives

$$\dot{\lambda} = \frac{\underset{\approx}{B}:\dot{\underset{\approx}{\sigma}}^{\text{trial}}}{\underset{\approx}{A}:\underset{\approx}{B} - \underset{\approx}{G}:\underset{\approx}{M}}. \tag{104}$$

Substituting this result back into Eq. (100) gives

$$\boxed{\dot{\underset{\approx}{\sigma}} = \begin{cases} h(\dot{\underset{\approx}{\sigma}}^{\text{trial}}) & \text{during continued yielding} \\ \dot{\underset{\approx}{\sigma}}^{\text{trial}} & \text{during elastic intervals} \end{cases}}, \tag{105}$$

where the operation h is defined

$$h(\underset{\approx}{X}) = \underset{\approx}{X} - \frac{\underset{\approx}{A}(\underset{\approx}{B}:\underset{\approx}{X})}{\underset{\approx}{A}:\underset{\approx}{B} - \underset{\approx}{G}:\underset{\approx}{M}} = h_{\underset{\approx}{A},\,\underset{\approx}{B},\,\underset{\approx}{G},\,\underset{\approx}{M}}(\underset{\approx}{X}). \tag{106}$$

This function is not a projector, but it is linear. Consequently, the numerical derivation on page 19 remains valid up through Eq. (48) if the function P is replaced with h. In particular, the generalization of Eq. (48) becomes

$$\underset{\approx}{\sigma}^{\text{new}} = \underset{\approx}{\sigma}^{\text{trial}} - \dot{\underset{\approx}{\sigma}}^{\text{trial}}\Delta t^{p} + h(\dot{\underset{\approx}{\sigma}}^{\text{trial}}\Delta t^{p}). \tag{107}$$

Even though this hardening solution depends the new function h, the original projection function P of Eq. (38) is nevertheless still well-defined. In fact, the new function h is related to the projector P by

$$h(\underset{\approx}{X}) = P(\underset{\approx}{X}) + \gamma\,\underset{\approx}{A}(\underset{\approx}{B}:\underset{\approx}{X}), \tag{108}$$

where γ is a scalar-valued function of the state tensors $\underset{\approx}{A}$, $\underset{\approx}{B}$, $\underset{\approx}{G}$, and $\underset{\approx}{M}$. Thankfully, the precise expression for γ is not needed for the ensuing discussion, but interested readers can readily verify that it is given by

$$\gamma = \frac{1}{\underset{\approx}{A}:\underset{\approx}{B} - \underset{\approx}{G}:\underset{\approx}{M}} - \frac{1}{\underset{\approx}{A}:\underset{\approx}{B}}. \tag{109}$$

Furthermore, if the evolution laws are presented in the form of Eq. (76), then

$$-\underset{\approx}{G}:\underset{\approx}{M} = \sum_{k}\frac{\partial f}{\partial \eta_{k}}\,m_{k} \tag{110}$$

Operating on Eq. (108) by P (and, as always, using linearity of P) gives

$$P(h(\underset{\approx}{X})) = P(P(\underset{\approx}{X})) + \gamma P(\underset{\approx}{A})(\underset{\approx}{B}:\underset{\approx}{X}) \tag{111}$$

Recalling that $P(P(\underset{\approx}{X})) = P(\underset{\approx}{X})$ and noting that $P(\underset{\approx}{A}) = \underset{\approx}{0}$, this simplifies to a useful identity that relates the projector P to the nonprojector h:

$$\boxed{P(h(\underset{\approx}{X})) = P(\underset{\approx}{X}) \text{ for any second-order tensor } \underset{\approx}{X}.} \tag{112}$$

Operating on Eq. (107) by P and applying (112) gives an equation identical to Eq. (49) from the nonhardening analysis. Consequently, *the final result is unchanged for the hardening (or softening) case!* Namely,

$$P(\underset{\approx}{\sigma}^{\text{new}}) = P(\underset{\approx}{\sigma}^{\text{trial}}) \tag{113}$$

In other words, the updated stress is still given by a projection of the trial stress back to the yield surface despite the fact that the stress *rate* is no longer a projection of the trial stress rate. Obviously, after an interval of plastic loading, the yield surface will have changed, and it is this new yield surface that would be used as the target in the projection. First Eq. (104) should be solved for the plasticity parameter $\dot{\lambda}$, after which the evolution equations may be integrated for updated values of the internal state variables at the end of the step. Finally the stress would be found by using a secant (or other) solver to find β in the generalization of Eq. (52); namely

$$\boxed{f(\underset{\approx}{\sigma}^{\text{trial}} + \beta \underset{\approx}{A},\ \ \eta_1^{\text{new}}, \eta_2^{\text{new}}, \ldots) = 0}. \tag{114}$$

Once β is known, the updated stress is simply

$$\underset{\approx}{\sigma}^{\text{new}} = \underset{\approx}{\sigma}^{\text{trial}} + \beta \underset{\approx}{A} \tag{115}$$

As mentioned in the context of nonhardening plasticity, a secant solver may not be necessary if the yield function is simple enough to solve Eq. (114) analytically. To summarize: the effect of hardening or softening is to move the target projection surface, but the projection to the *updated* surface still holds. The effect of elastic-plastic coupling is to alter the projection *direction*. Consequently, the analyst must be careful not to mistake the effect of elastic-plastic coupling as a sign of non-associative flow.

Closing comments

The purpose of this technical note was to provide a geometric interpretation of conventional return algorithms for nonhardening plasticity, with a brief explanation of why these algorithms continue to work even when hardening and/or softening and/or elastic-plastic coupling are allowed. It's important to recognize the distinction between projection of the stress *rate*, which holds only for stationary (nonhardening) yield surfaces, and projection of the trial stress, which is valid even for hardening or softening yield surfaces. The solution algorithm for *both hardening and nonhardening* yield surfaces reduces to a simple task of using the projection direction tensor $\underset{\sim}{A}$ to place the elastic trial stress on the yield surface.

Another important point of this discussion is that the direction used to project the trial stress back to the yield surface is not necessarily proportional to the direction of the plastic strain rate. An associated flow rule can have a nonassociated return direction and vice versa.

The radial and oblique return algorithms are rigorous direct consequences of the governing equations (22) through (25). There is *nothing* in these derivations that requires any attention to positive dissipation, convexity of the yield surface, or plastic stability. Such concerns merely dictate appropriate choices for quantities that have been herein presumed known.

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02.7.30

APPENDIX 1: Spectrum of the *isotropic* elastic stiffness.

A material is elastic if stress $\underset{\approx}{\sigma}$ can be written as a proper function of the strain $\underset{\approx}{\varepsilon}$. Consequently, the stress-strain path is the same for both loading and unloading.* Even if the elasticity relationship is nonlinear, the chain rule shows that the stress *rate* is linear with respect to the strain rate:

$$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}}, \qquad \text{where} \qquad E_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \tag{116}$$

The fourth-order tensor $\underset{\approx\approx}{E}$ is called the elastic *tangent* modulus. It is analogous to the local tangent to the nonlinear elastic stress-strain curve for uniaxial deformations. For isotropic elasticity, the stiffness tensor $\underset{\approx\approx}{E}$ is expressible in the form [2]

$$E_{ijkl} = (K - \tfrac{2}{3}G)(\delta_{ij}\delta_{kl}) + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{117}$$

where K is the elastic bulk modulus, G is the shear modulus, and δ_{ij} is the Kronecker delta (equal to 1 if $i = j$ and zero otherwise).

When the isotropic stiffness of Eq. (117) operates on the strain rate in Eq. (116), the resulting formula is known as Hooke's law:

$$\dot{\underset{\approx}{\sigma}} = 3K\dot{\underset{\approx}{\varepsilon}}^{i} + 2G\dot{\underset{\approx}{\varepsilon}}^{\mathrm{d}}, \tag{118}$$

The tensors $\dot{\underset{\approx}{\varepsilon}}^{i}$ and $\dot{\underset{\approx}{\varepsilon}}^{\mathrm{d}}$ are, respectively, the isotropic and deviatoric parts of $\dot{\underset{\approx}{\varepsilon}}$, defined

$$\dot{\underset{\approx}{\varepsilon}}^{i} = \tfrac{1}{3}\mathrm{trace}(\dot{\underset{\approx}{\varepsilon}})\underset{\approx}{I} \quad \text{and} \quad \dot{\underset{\approx}{\varepsilon}}^{\mathrm{d}} = \dot{\underset{\approx}{\varepsilon}} - \dot{\underset{\approx}{\varepsilon}}^{i}, \quad \text{where } \mathrm{trace}(\dot{\underset{\approx}{\varepsilon}}) = \sum_{k=1}^{3} \dot{\varepsilon}_{kk} \tag{119}$$

Eq. (118) shows that, for isotropic elasticity, the stress rate is a linear combination of the isotropic and deviatoric strain rates. Whenever the strain rate is traceless (*i.e.*, whenever it is already deviatoric) then $\dot{\underset{\approx}{\sigma}} = 2G\dot{\underset{\approx}{\varepsilon}}$. Geometrically, this means that, whenever the strain rate is deviatoric, the stress rate differs from the strain rate by only a scalar multiple $2G$. Referring to Eq. (116), this means that

$$\underset{\approx\approx}{E} : \underset{\approx}{A} = 2G\underset{\approx}{A} \quad \text{for } \textit{any} \text{ symmetric deviatoric tensor } \underset{\approx}{A}. \tag{120}$$

Hence, for isotropic elasticity, any symmetric deviatoric tensor is an "eigentensor" of $\underset{\approx\approx}{E}$, and the associated eigenvalue is $2G$. Similarly, any isotropic tensor (*i.e.*, any tensor of the form $\alpha\underset{\approx}{I}$ for some scalar α) is an eigentensor with the eigenvalue equal to $3K$.

*For any material, elastic or not, subjected to any deformation, stress and strain can be written parametrically as functions of time. Therefore stress is often regarded as an *implicit* function of strain. However, to be a *true* or *proper* function of strain, the stress must be uniquely determined by the strain regardless of the path through time (*i.e.*, proper functions must be single-valued).

APPENDIX 2: Further explanation of the plasticity equations.

This appendix provides brief derivations or explanations of the governing equations (22) through (25). The equation $f(\underset{\sim}{\sigma}) = 0$ describes the yield surface in 6D stress space. The yield surface is recognized as nonhardening because it is strictly a function of stress — it does not change size or shape with time.* For continued yield, the stress state must remain on the yield surface. In other words, $\dot{f}=0$ for continued yield. Applying the chain rule gives Eq. (22). Namely,

$$\dot{f} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}\dot{\sigma}_{ij} = 0 . \tag{121}$$

The geometrical interpretation of Eq. (22) is that the stress rate $\dot{\underset{\sim}{\sigma}}$ must be tangent to the yield surface. Since the gradient $\underset{\sim}{B}$ of the yield function is normal to the yield surface, the stress rate must be "perpendicular" to $\underset{\sim}{B}$. This restriction follows from the nonhardening assumption that the yield function depends *only* on the stress $\underset{\sim}{\sigma}$ and not on any other internal state variables that could cause the yield surface to expand or contract, which would give the stress rate a normal component in order to "keep up." This effect is discussed separately on page 26.

Equation (23) expresses the typical decomposition of the strain rate $\dot{\underset{\sim}{\varepsilon}}$ into recoverable elastic part $\dot{\underset{\sim}{\varepsilon}}^e$ plus a permanent plastic part $\dot{\underset{\sim}{\varepsilon}}^p$. Heuristically, the plastic strain may be regarded as the permanent strain that would remain if all stresses were released.

Equation (24) states that the magnitude $\dot{\lambda}$ of the plastic strain rate $\dot{\underset{\sim}{\varepsilon}}^p$ is not known, but its *direction* $\underset{\sim}{M}$ is presumed known based on other physical arguments. Importantly, this direction depends only on the material state. In other words, all rate dependence (if any) is reflected through the magnitude $\dot{\lambda}$ of the plastic strain rate. The *direction* of the plastic strain rate is presumed rate independent. Whenever the plastic strain rate is normal to the yield surface (*i.e.*, if $\underset{\sim}{M}=\alpha\underset{\sim}{B}$ for some positive scalar α), then the flow rule is said to be "associative".

Equation (25) states that the stress rate is linearly related to the *elastic* part of the strain rate. The component form of (25) is

$$\dot{\sigma}_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} E_{ijkl}\dot{\varepsilon}^e_{kl} \tag{122}$$

If the material is isotropic, this equation is merely the elastic expression of Hooke's law of Eq. (118) in rate form.

*Hardening and softening yield surfaces that *do* vary in time are discussed on page 26.

Importance of the yield function sign convention. If a material is plastically isotropic, its yield function satisfies the condition that $f(\underset{\approx}{Q}^T \bullet \underset{\approx}{\sigma} \bullet \underset{\approx}{Q}) = f(\underset{\approx}{\sigma})$ for all orthogonal tensors $\underset{\approx}{Q}$. This implies that the yield function can be written as a function of invariants of stress. One of the simplest isotropic yield functions is that of Tresca which hypothesizes that yield commences when the largest shear stress reaches a critical value, k. If σ_{max} and σ_{min} are respectively the largest and smallest principal values of stress, then the largest shear is $\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min})$ and occurs on a plane whose normal bisects two associated principal directions of stress. If the three principal stresses are known but have not yet been ordered, the Tresca criterion can be applied by simply considering the largest (in absolute value) of the three possible stress differences. In other words, the Tresca yield function may be written

$$f(\underset{\approx}{\sigma}) = \max((\sigma_1 - \sigma_2)^2, (\sigma_2 - \sigma_3)^2, (\sigma_3 - \sigma_1)^2) - 4k^2 \tag{123}$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ are the principal stresses. This is a properly defined yield function because it satisfies the essential sign convention properties:

$$f = 0 \text{ if and only if the stress is on the yield surface} \tag{124a}$$

$$f < 0 \text{ if and only if the stress is inside the yield surface.} \tag{124b}$$

The choice of the yield function is not unique. Many other functions can be constructed such that the above two conditions hold. Unfortunately, however, many authors* will wrongly claim that an alternative form for the Tresca yield function is

$$f^*(\underset{\approx}{\sigma}) = [(\sigma_1 - \sigma_2)^2 - 4k^2][(\sigma_2 - \sigma_3)^2 - 4k^2][(\sigma_3 - \sigma_1)^2 - 4k^2] \tag{125}$$

It is straightforward to verify that this function may be written in terms of the standard invariants, $J_2 = \frac{1}{2}\left\|\underset{\approx}{S}\right\|$ and $J_3 = \det \underset{\approx}{S}$, of the stress deviator $\underset{\approx}{S}$:

$$f^*(\underset{\approx}{\sigma}) = 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 \tag{126}$$

Thus, the function f^* has an intoxicating appeal because its value may be computed without an eigenvalue analysis of the stress. The only problem is that f^* is not a valid yield function! The properties of Eq. (124) are satisfied in only one direction, not both. The function f^* has properties

$$\text{If a stress is on the yield surface, then } f^* = 0. \tag{127a}$$

$$\text{If a stress is within the yield surface, then } f^* < 0. \tag{127b}$$

However, the converses of both statements are false and therefore the sign requirements of Eqs. (124) do not hold! The proof by counterexample is trivial. Consider $\sigma_1 = \sigma_2 = 3k$ and $\sigma_3 = 0$. Applying the *valid* yield function of Eq. (123) gives $f > 0$, indicating that this stress state lies outside the yield surface. Applying the *invalid* yield function of Eq. (125) gives $f^* = -20k^2$, which (being negative) *falsely* indicates that this stress state is *inside* the yield surface. Conclusion: the function f^* is inadmissible as the sole means of determining whether a stress state is below yield.

*and even respected plasticity texts [1].

The situation is not too hard to rectify. The basic idea is to *first* check whether the stress state lies near the yield surface by checking whether it lies within the inscribed and circumscribed Von Mises yield surfaces that bound the Tresca hexagon. Specifically,

$$\begin{aligned}&\text{If } J_2 < k^2, \text{ then the stress is inside the yield surface,}\\ &\text{else if } J_2 > 4k^2/3, \text{ then the stress is outside the yield surface,}\\ &\text{else check the sign of Eq. (126).}\end{aligned} \tag{128}$$

This corresponds to the following yield function

$$f(\underset{\approx}{\sigma}) = \begin{cases} J_2 - 4k^2/3 \text{ if } J_2 > 4k^2/3 & \text{(129a)} \\ 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 \text{ otherwise} & \text{(129b)} \end{cases}$$

Elastic potential

Quite often, the stress is presumed derivable from an elastic potential $W = W(\underset{\approx}{\varepsilon}^e)$ such that

$$\underset{\approx}{\sigma} = \frac{dW}{d\underset{\approx}{\varepsilon}^e} = \frac{\partial W(\underset{\approx}{\varepsilon} - \underset{\approx}{\varepsilon}^p)}{\partial \underset{\approx}{\varepsilon}} \tag{130}$$

The elastic stiffness is defined

$$\underset{\approx\approx}{E} = \frac{d\underset{\approx}{\sigma}}{d\underset{\approx}{\varepsilon}^e} = \frac{d^2W}{d\underset{\approx}{\varepsilon}^e d\underset{\approx}{\varepsilon}^e}, \text{ or } E_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon^e_{ij} \partial \varepsilon^e_{kl}} \tag{131}$$

If the stiffness is constant, the elastic potential function is expressible as

$$W = \tfrac{1}{2}\underset{\approx}{\varepsilon}^e : \underset{\approx\approx}{E} : \underset{\approx}{\varepsilon}^e \tag{132}$$

ISV potential

Hardening or softening yield functions are of the form $f(\underset{\approx}{\sigma}, \eta_1, \eta_2, \ldots)$. Simo and Hughes [3] define the flow rule to be "associative" if the plastic strain rate is normal to the yield surface in stress space

$$\dot{\underset{\approx}{\varepsilon}}^p = \gamma \frac{\partial f}{\partial \underset{\approx}{\sigma}} = \gamma \underset{\approx}{\boldsymbol{B}} \tag{133}$$

Recalling that we write

$$\dot{\underset{\approx}{\varepsilon}}^p = \dot{\lambda} \underset{\approx}{\boldsymbol{M}}, \tag{134}$$

where $\underset{\approx}{\boldsymbol{M}}$ is a unit tensor,

Stress space associativity is equivalent to assuming that

$$\underset{\approx}{\boldsymbol{M}} = \frac{\underset{\approx}{\boldsymbol{B}}}{\left\|\underset{\approx}{\boldsymbol{B}}\right\|} \quad \text{and} \quad \dot{\lambda} = \gamma\left\|\underset{\approx}{\boldsymbol{B}}\right\| \tag{135}$$

Simo and Hughes also extend the idea of associativity to include the following assumption about the evolution of the internal state variables. Specifically, we will say that the model is "fully associative" if there exist coefficients D_{kj} (called plastic moduli) such that

$$\dot{\eta}_k = -\gamma \sum_{j=1}^{N^{\text{isv}}} D_{kj}\frac{\partial f}{\partial \eta_j} = -\dot{\lambda} \sum_{j=1}^{N^{\text{isv}}} d_{kj}\frac{\partial f}{\partial \eta_j} \tag{136}$$

Here d_{kj} are just an alternative set of plastic moduli defined in a manner compatible with $\dot{\lambda}$. Namely,

$$d_{kj} = \frac{D_{kj}}{\left\|\underset{\approx}{\boldsymbol{B}}\right\|} \tag{137}$$

Equation (136) can be written

$$\gamma\frac{\partial f}{\partial \eta_k} = -\sum_{j=1}^{N^{\text{isv}}} D_{kj}^{-1}\dot{\eta}_j \tag{138}$$

It is natural to define "conjugate" internal state variables A_k by

$$A_k = -\sum_{j=1}^{N^{\text{isv}}} D_{kj}^{-1}\eta_j \tag{139}$$

Thus, if the plastic moduli are constant, then Eq. (138) can be written

$$\dot{A}_k = \gamma\beta_k \text{ where } \beta_k = \frac{\partial f}{\partial \eta_j} \tag{140}$$

Alternatively, using $\dot{\lambda}$ and d_{kj}, we note that equation (136) can be written

$$\dot{\lambda}\frac{\partial f}{\partial \eta_k} = -\sum_{j=1}^{N^{\text{isv}}} d_{kj}^{-1}\dot{\eta}_j \tag{141}$$

It is natural to define "conjugate" internal state variables a_k by

$$a_k = -\sum_{j=1}^{N^{\text{isv}}} d_{kj}^{-1}\eta_j = \frac{A_k}{\left\|\underset{\approx}{\boldsymbol{B}}\right\|} \tag{142}$$

Thus, if the plastic moduli are constant, then Eq. (138) can be written

$$\dot{a}_k = \dot{\lambda}\beta_k, \text{ where } \beta_k = \frac{\partial f}{\partial \eta_j} \tag{143}$$

Note that this result is analogous to Eq. (133). Simo and Hughes therefore postulate the existence of a plastic potential H such that

$$\eta_k = -\frac{\partial H}{\partial A_k} \tag{144}$$

If the plastic moduli are constant, then

$$H = \tfrac{1}{2}\sum_{k=1}^{N^{\text{isv}}}\sum_{j=1}^{N^{\text{isv}}} A_k D_{kj} A_j \tag{145}$$

Example: porosity. For a plastically incompressible matrix material, we know that the unstressed porosity ϕ must evolve according to

$$\dot{\phi} = (1-\phi)\text{tr}\dot{\underset{\approx}{\varepsilon}}^p \tag{146}$$

Alternatively, the porosity evolution equation may be written in terms of the commonly-used distention α defined by $\alpha = 1/(1-\phi)$. Then the porosity evolution equation is

$$\dot{\alpha} = \alpha\text{tr}\dot{\underset{\approx}{\varepsilon}}^p \tag{147}$$

We can alternatively define

$$\xi \equiv \ln\alpha \tag{148}$$

Then the evolution equation for this alternative porosity measure is

$$\dot{\xi} = \text{tr}\dot{\underset{\approx}{\varepsilon}}^p \tag{149}$$

Note that all three porosity measures (ϕ, α, and ξ) represent *the same* internal state variable. Knowing one gives values for the other. To analyze all three choices (as well as other unstated choices) for the porosity variable, let η denote some unique measure of porosity. We will assume that there exists a function

$$\eta = \bar{\eta}(\xi) \tag{150}$$

that relates the porosity measure to the porosity measure ξ. Then Eq. (149) implies that η is governed by

$$\dot{\eta} = \frac{d\eta}{d\xi}\dot{\xi} = \bar{\eta}'(\xi)\text{tr}\dot{\underset{\approx}{\varepsilon}}^p, \text{ where } \bar{\eta}' = \frac{d\bar{\eta}(\xi)}{d\xi} \tag{151}$$

For example,

$$\begin{aligned}
&\text{if } \eta = \phi, \text{ then } \bar{\eta}(\xi) = 1 - e^{-\xi} \text{ and } \bar{\eta}' = e^{-\xi} = 1-\phi \\
&\text{if } \eta = \alpha, \text{ then } \bar{\eta}(\xi) = e^{\xi} \text{ and } \bar{\eta}' = e^{\xi} = \alpha \\
&\text{if } \eta = \xi, \text{ then } \bar{\eta}(\xi) = \xi \text{ and } \bar{\eta}' = 1
\end{aligned} \tag{152}$$

The porosity may be regarded as an internal state variable. Now we are going to explore Simo and Hughes idea of the "plastic" modulus for our generalized porosity measure η. Keep in mind that an application will use one — *and only one* — measure of porosity. Our purpose here is to illustrate how the *choice* of porosity measure can have an impact on the simplicity of the calculations.

Recall that the plastic strain rate may be written as

$$\dot{\underset{\approx}{\varepsilon}}^p = \gamma \frac{\partial f}{\partial \underset{\approx}{\sigma}} \tag{153}$$

Then the three evolution equation becomes

$$\dot{\eta} = \gamma \left[\bar{\eta}' \, \mathrm{tr} \frac{\partial f}{\partial \underset{\approx}{\sigma}} \right] \tag{154}$$

For this special case of a *single* internal state variable, the summation in Eq. (136) now ranges over only one term. In other words, there is only one η_k, so the D_{kj} matrix is just a 1×1 matrix, making it simply a single scalar D. Depending on the *single* choice for the porosity measure, Eq. (136) becomes

$$\dot{\eta} = -\gamma D \frac{\partial f}{\partial \eta} \tag{155}$$

Equating the last two equation gives

$$D \frac{\partial f}{\partial \eta} = -\bar{\eta}' \, \mathrm{tr} \frac{\partial f}{\partial \underset{\approx}{\sigma}} \tag{156}$$

Using subscripts to explicitly show what is being held constant, this equation may be written

$$D \left(\frac{\partial f}{\partial \eta} \right)_{\underset{\approx}{\sigma}} = -\bar{\eta}' \, \mathrm{tr} \left(\frac{\partial f}{\partial \underset{\approx}{\sigma}} \right)_{\eta} \tag{157}$$

Since η is the only internal state variable, we know that the yield function is of the form $f = f(\underset{\approx}{\sigma}, \eta)$. Without loss in generality, we may alternatively assert that $f = f(\underset{\approx}{\boldsymbol{S}}, p, \eta)$, where $\underset{\approx}{\boldsymbol{S}}$ is the stress deviator and p is the pressure. Then, by the chain rule,*

$$\mathrm{tr} \left(\frac{\partial f}{\partial \underset{\approx}{\sigma}} \right)_{\eta} = -\left(\frac{\partial f}{\partial p} \right)_{\underset{\approx}{S}, \eta} \tag{158}$$

Now, observe that

$$\left(\frac{\partial f}{\partial \eta} \right)_{\underset{\approx}{\sigma}} = \left(\frac{\partial f}{\partial \eta} \right)_{\underset{\approx}{S}, p} \tag{159}$$

By a well-known identity from multivariable calculus, we note that

$$\frac{\left(\frac{\partial f}{\partial \eta} \right)_{\underset{\approx}{S}, p}}{\left(\frac{\partial f}{\partial p} \right)_{\underset{\approx}{S}, \eta}} = -\left(\frac{\partial p}{\partial \eta} \right)_{f, \underset{\approx}{S}} \tag{160}$$

Substituting Eq. (158) into Eq. (157) and using the identity (160) gives

$$D \left(\frac{\partial p}{\partial \eta} \right)_{f, \underset{\approx}{S}} = -\bar{\eta}' \tag{161}$$

When solving this partial differential equation, it is important to realize that any integration constants that appear will actually be functions of the constant stress deviator $\underset{\approx}{\boldsymbol{S}}$. Suppose that "crush curve" experiments are available that provide the porosity as a function of

*Incidentally, this operation is a good example of a danger with indicial notation. Specifically, $\mathrm{tr}(\partial f / \partial \underset{\approx}{\sigma})$ is not the same thing as $\partial f / \partial (\mathrm{tr} \underset{\approx}{\sigma})$, so it should not be written in indicial notation as $\mathrm{tr}(\partial f / \partial \sigma_{kk})$ because such an expression is ambiguous.

the applied pressure, for various (constant) values of the stress deviator. Such curves are often called p-α curves. By the substitution $\alpha = e^{\xi}$, these curves can be readily converted into p-ξ curves. An experimentally measured $p = p(\xi, \underset{\sim}{\boldsymbol{S}})$ function applies during crush (i.e., when $f=0$). Therefore, we can define

$$g(\xi, \underset{\sim}{\boldsymbol{S}}) \equiv -\left(\frac{\partial p}{\partial \xi}\right)_{\underset{\sim}{S}} \tag{162}$$

Thus, the function $g = g(\xi, \underset{\sim}{\boldsymbol{S}})$ may be regarded as a known property of the material. We have used the negative sign because a decrease in porosity ξ normally results from an increase in pressure (though the opposite is true in tension). By the chain rule, we have

$$\left(\frac{\partial p}{\partial \eta}\right)_{\underset{\sim}{S}} = \left(\frac{\partial p}{\partial \xi}\right)_{\underset{\sim}{S}} \frac{d\xi}{d\eta} = -\frac{g}{\bar{\eta}'} \tag{163}$$

So Eq. (161) becomes

$$D\frac{g}{\bar{\eta}'} = \bar{\eta}' \tag{164}$$

or

$$\bar{\eta}' = \sqrt{Dg} \tag{165}$$

or, showing the independent variables,

$$\bar{\eta}'(\xi) = \sqrt{Dg(\xi, \underset{\sim}{\boldsymbol{S}})} \tag{166}$$

In order for this to hold, we note that the plastic modulus D must generally depend on the stress deviator. If the function $g(\xi, \underset{\sim}{\boldsymbol{S}})$ is approximated to be independent of the stress deviator, then the above equation may be integrated to obtain the "natural" measure of porosity for which the plastic modulus is constant. Alternatively, if η is *assumed* to be the natural porosity measure, then D is constant and Eq. (166) implies the crush curve. Namely

$$g(\xi, \underset{\sim}{\boldsymbol{S}}) = \frac{[\bar{\eta}'(\xi)]^2}{D} \tag{167}$$

or

$$p - p_o = -\frac{1}{D}\int_{\xi_o}^{\xi} [\bar{\eta}'(\xi^*)]^2 d\xi^* \tag{168}$$

If, for example, ξ itself is presumed to be the natural porosity measure, then

$$p - p_o = -\frac{1}{D}(\xi - \xi_o) \tag{169}$$

or, writing $\xi = \ln\alpha$, the crush curve becomes

$$\frac{\alpha}{\alpha_o} = e^{-D(p - p_o)} \tag{170}$$

The integration constants α_o and p_o depend on the shear stress.

Plastic dissipation

Simo and Hughes *define* the plastic dissipation to be

$$D^p = \underset{\approx}{\sigma} : \dot{\underset{\approx}{\varepsilon}}^p + \sum_{k=1}^{N^{\text{isv}}} \eta_k \dot{A}_k \tag{171}$$

where A_k are the conjugate "strains" associated with the internal state variables; namely

$$A_k = -\sum_{j=1}^{N^{\text{isv}}} D_{kj}^{-1} \eta_j \tag{172}$$

Example: porosity. Consider a single porosity measure η used in the previous example. Suppose that D is constant. Then

$$D^p = \underset{\approx}{\sigma} : \dot{\underset{\approx}{\varepsilon}}^p + \eta \dot{A} \tag{173}$$

where A is the conjugate "strains" associated with the internal state variables; namely

$$A = -\frac{\eta}{D} \tag{174}$$

Using Eq. (166) gives

$$\eta = \int_{\xi_o}^{\xi} \sqrt{D g(\xi^*, \underset{\approx}{\boldsymbol{S}})}\, d\xi^* \tag{175}$$

Therefore

$$A = \int_{\xi_o}^{\xi} \sqrt{\frac{g(\xi^*, \underset{\approx}{\boldsymbol{S}})}{D}}\, d\xi^* \tag{176}$$

And

$$\dot{A} = -\frac{\dot{\eta}}{D} = -\dot{\xi}\sqrt{\frac{g(\xi, \underset{\approx}{\boldsymbol{S}})}{D}} \tag{177}$$

APPENDIX 3: Pressure-dependent Von Mises (J2) yield

The traditional Von Mises criterion is often modified to include pressure dependence by allowing the Von Mises yield stress to vary with pressure. If the material is plastically incompressible, then the solution algorithm is not significantly different from the solution method described on page 24 for the traditional pressure-independent Von Mises model. Specifically, if the material is plastically incompressible, then the updated stress is obtained by merely scaling down the magnitude of the trial stress *deviator*, keeping the trial pressure as the final pressure. As explained below, this so-called "Prandtl" solution implicitly corresponds to a nonassociative flow rule. Furthermore, such an approach is grossly inappropriate for porous materials because porous materials exhibit large amounts of plastic volume change.

Unfortunately, if the plastic strain rate is allowed to have an isotropic part, correctly accounting for the pressure dependence is quite complicated. This appendix shows how the choice of distortional and volumetric stress measures can have profound impact on the solution scheme. This appendix closes with a detailed algorithm for updating the stress for any plastic strain direction.

Mathematical/geometrical preliminaries. To begin the discussion of pressure sensitive Von Mises yield models, it is useful to demonstrate that breaking up a tensor into its isotropic and deviatoric parts is a lot like decomposing an ordinary vector into its parts that are parallel and perpendicular to a fixed unit vector.

Figure 6 shows an ordinary 3D vector $\underset{\sim}{x}$ decomposed into parts that are parallel and perpendicular to a unit vector $\underset{\sim}{n}$. The part of $\underset{\sim}{x}$ in the direction of $\underset{\sim}{n}$ is just

$$x_n \underset{\sim}{n}, \quad \text{where} \quad x_n = \underset{\sim}{x} \bullet \underset{\sim}{n} \tag{178}$$

The part of $\underset{\sim}{x}$ perpendicular to $\underset{\sim}{n}$ is then simply

$$\underset{\sim}{s} = \underset{\sim}{x} - x_n \underset{\sim}{n} \tag{179}$$

Let x_s denote the magnitude of $\underset{\sim}{s}$, and let $\hat{\underset{\sim}{s}}$ be the unit vector in the direction of $\underset{\sim}{s}$. Then we can write the above expression as

$$\underset{\sim}{x} = x_n \underset{\sim}{n} + x_s \hat{\underset{\sim}{s}} \tag{180}$$

In this form, x_n and x_2 can be viewed as the components of $\underset{\sim}{x}$ with respect to the unit base vectors, $\underset{\sim}{n}$ and $\hat{\underset{\sim}{s}}$, in the plane. If a physical phenomenon of interest is occurring exclusively in the plane, it can be analyzed in two dimensions rather than three dimensions.

Suppose that $\underset{\sim}{u}$ is some other vector in the *same* plane, then it can be written as a linear combination of the *same* base vectors:

$$\underset{\sim}{u} = u_n \underset{\sim}{n} + u_s \hat{\underset{\sim}{s}}, \tag{181}$$

where

$$u_n = \underset{\sim}{u} \bullet \underset{\sim}{n} \qquad \text{and} \qquad u_s = \underset{\sim}{u} \bullet \hat{\underset{\sim}{s}} \tag{182}$$

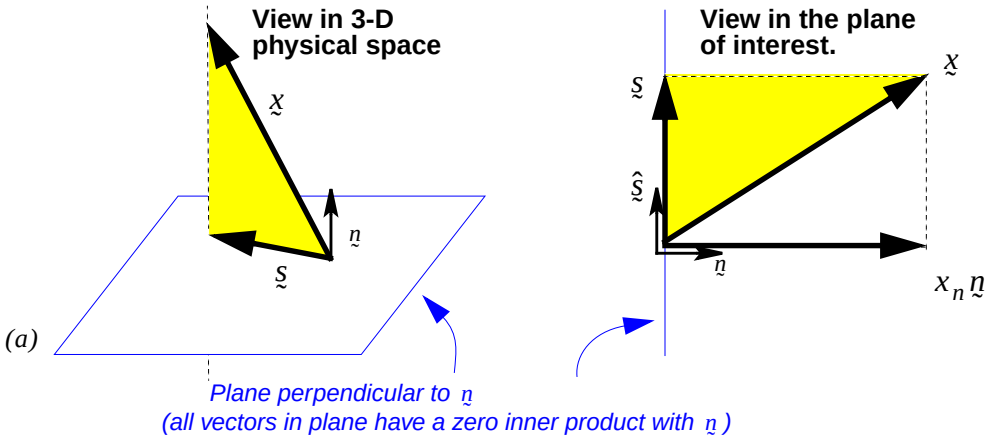

Figure 6. Orthogonal projection of a vector. (a) full three-dimensional view. (b) view of the same configuration as seen from an optimal perspective where all relevant vectors are in the observer's plane and the unit vector $\underset{\sim}{n}$ points to the observer's right.

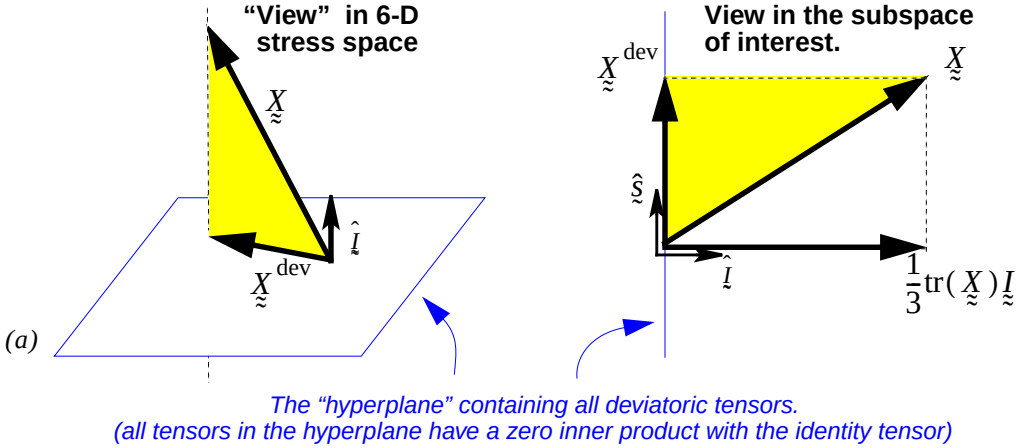

Figure 7. Decomposition of a tensor into isotropic and deviatoric parts. (a) conceptual "view" in stress space. (b) view of the same configuration as seen from an optimal perspective where the tensor is broken up into its isotropic and deviatoric parts.

Now consider the analogous *tensor* operation in which a tensor is decomposed into its isotropic and deviatoric parts. The analogy with the ordinary vector decomposition is seen by comparing Figures 6 and 7. A tensor is said to be isotropic if it is proportional to the identity tensor $\underset{\approx}{I}$. Thus, the tensor $\underset{\approx}{I}$ serves as a *basis* for all isotropic tensors. Unfortunately, $\underset{\approx}{I}$ is not a unit tensor; its magnitude is

$$\left|\left|\underset{\approx}{I}\right|\right| = \sqrt{\underset{\approx}{I} : \underset{\approx}{I}} = \sqrt{1+1+1} = \sqrt{3} \tag{183}$$

We introduce a unit tensor in the direction of $\underset{\approx}{I}$, defined

$$\hat{\underset{\approx}{I}} \equiv \frac{\underset{\approx}{I}}{\left|\left|\underset{\approx}{I}\right|\right|} = \frac{1}{\sqrt{3}}\underset{\approx}{I} \tag{184}$$

The unit tensor $\hat{\underset{\approx}{I}}$ in Fig. 7 is analogous to the unit vector $\underset{\sim}{n}$ in Fig. 6. The isotropic part of a tensor $\underset{\approx}{X}$ is defined

$$\underset{\approx}{X}^{\text{iso}} \equiv \frac{1}{3}\text{tr}(\underset{\approx}{X})\underset{\approx}{I} \tag{185}$$

It's easily verified that this can be written

$$\underset{\approx}{X}^{\text{iso}} \equiv X_p \hat{\underset{\approx}{I}}, \qquad \text{where} \qquad X_p \equiv \underset{\approx}{X} : \hat{\underset{\approx}{I}} \tag{186}$$

We use the subscript "p" because, for tensors, the isotropic part is often related in some way to pressure. Recall that the double dot product for tensors is analogous to the single dot product for ordinary vectors. Hence Eq. (186) is analogous to Eq. (178). Finding the isotropic part of a tensor is exactly like orthogonally projecting a vector to its part parallel to a given unit vector. The isotropic part of a tensor is the part of the tensor that is "parallel" to the identity tensor!

The "deviatoric" part of a tensor $\underset{\approx}{X}$ is defined

$$\underset{\approx}{X}^{\text{dev}} \equiv \underset{\approx}{X} - \underset{\approx}{X}^{\text{iso}} \tag{187}$$

This equation is analogous to Eq. (179). We now introduce a unit tensor in the direction of $\underset{\approx}{X}^{\text{dev}}$:

$$\hat{\underset{\approx}{S}} \equiv \frac{\underset{\approx}{X}^{\text{dev}}}{\left|\left|\underset{\approx}{X}^{\text{dev}}\right|\right|} \tag{188}$$

If we denote the magnitude of $\underset{\approx}{X}^{\text{dev}}$ by $\tilde{X}_s$, then the tensor $\underset{\approx}{X}$ can be written

$$\underset{\approx}{X} = X_p \hat{\underset{\approx}{I}} + X_s \hat{\underset{\approx}{S}} \tag{189}$$

This equation is analogous to Eq. (180). Note that the tensors $\hat{\underset{\approx}{I}}$ and $\hat{\underset{\approx}{S}}$ form a basis for a two-dimensional subspace of nine-dimensional tensor space. This subspace is analogous to the plane containing the vectors $\underset{\sim}{n}$ and $\hat{\underset{\sim}{s}}$ in Fig. 6. Any tensor $\underset{\approx}{U}$ in this subspace can be written in the form

$$\underset{\approx}{U} = U_p \hat{\underset{\approx}{I}} + U_s \hat{\underset{\approx}{S}} \tag{190}$$

where

$$U_p \equiv \underset{\approx}{U} : \hat{\underset{\approx}{I}} = \frac{1}{\sqrt{3}}\text{tr}(\underset{\approx}{U})\underset{\approx}{I} \qquad \text{and} \qquad U_s \equiv \underset{\approx}{U} : \hat{\underset{\approx}{S}} = \left|\left|\underset{\approx}{U}^{\text{dev}}\right|\right| \tag{191}$$

These equations are analogous to Eqs. (181) and (182).

Note that $\tilde{U}_p$ can be regarded as a function of $\underset{\approx}{U}$. Likewise $\tilde{U}_s$ is also a function of $\underset{\approx}{U}$. Later on, we will need to know the derivatives of $\tilde{U}_p$ and $\tilde{U}_s$ with respect to $\underset{\approx}{U}$. In component form, note that

$$U_p = \tfrac{1}{\sqrt{3}}(U_{11} + U_{22} + U_{33}) \tag{192}$$

Consequently,

$$\frac{\partial U_p}{\partial U_{ij}} = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{\sqrt{3}} & \text{if } i = j \end{cases} = \tfrac{1}{\sqrt{3}}\delta_{ij} \tag{193}$$

In direct notation, we write this as

$$\frac{dU_p}{d\underset{\approx}{U}} = \tfrac{1}{\sqrt{3}}\underset{\approx}{I} = \hat{\underset{\approx}{I}} \tag{194}$$

Similarly, it is straightforward to show that

$$\frac{dU_s}{d\underset{\approx}{U}} = \hat{\underset{\approx}{S}} \tag{195}$$

Application to the stress tensor. The decomposition of the stress tensor $\underset{\approx}{\sigma}$ into deviatoric and isotropic parts is conventionally written

$$\underset{\approx}{\sigma} = -p\underset{\approx}{I} + \underset{\approx}{S} \tag{196}$$

where $p \equiv -\frac{1}{3}\text{tr}\underset{\approx}{\sigma}$ is the compressive pressure and $\underset{\approx}{S}$ is the stress deviator. Applying Eqs. (190) and (191) with $\underset{\approx}{U}$ replaced by $\underset{\approx}{\sigma}$ gives

$$\underset{\approx}{\sigma} = \sigma_p \hat{\underset{\approx}{I}} + \sigma_s \hat{\underset{\approx}{S}} \tag{197}$$

where

$$\sigma_p \equiv \underset{\approx}{\sigma} : \hat{\underset{\approx}{I}} = \tfrac{1}{\sqrt{3}}\text{tr}(\underset{\approx}{\sigma})\underset{\approx}{I} = -\sqrt{3}\, p \text{ , and}$$
$$\sigma_s \equiv \underset{\approx}{\sigma} : \hat{\underset{\approx}{S}} = \left\| \underset{\approx}{\sigma}^{\text{dev}} \right\| = \left\| \underset{\approx}{S} \right\| \tag{198}$$

As before,

$$\frac{d\sigma_p}{d\underset{\approx}{\sigma}} = \hat{\underset{\approx}{I}} \qquad \text{and} \qquad \frac{d\sigma_s}{d\underset{\approx}{\sigma}} = \hat{\underset{\approx}{S}} \tag{199}$$

Pressure-dependent Von Mises yield functions. (For plasticity applications involving a pressure-dependent Von Mises yield model, most of the tensors of interest will lie in the "plane" formed by the identity tensor and the stress deviator. Rather than explicitly giving a yield function, $f(\underset{\approx}{\sigma})$, most people define a stress state to be below yield if a measure of the shear stress is less

than some critical value that depends on pressure. Unfortunately, there are several different measures of the shear stress in common use. Likewise, there are various definitions for the pressure. To make the upcoming analysis useful to the largest audience of readers, we will define "engineering" stress measures:

$$\Sigma_p \equiv \gamma_p \sigma_p = -\sqrt{3}\gamma_p p \tag{200}$$

$$\Sigma_s \equiv \gamma_s \sigma_s = \gamma_s \left\| \underset{\approx}{S} \right\| \tag{201}$$

The scalar constants γ_s and γ_p are selected by the analyst to correspond to their preferred measures of shear and isotropic stresses. Typical/convenient choices for the deviatoric stress coefficient are

$$\text{if } \gamma_s = \frac{1}{\sqrt{2}} \text{ then } \Sigma_s \text{ is the so-called effective shear stress} \tag{202}$$

$$\text{if } \gamma_s = \sqrt{\frac{2}{3}} \text{ then } \Sigma_s \text{ is the effective (uniaxial) stress.} \tag{203}$$

$$\text{if } \gamma_s = 1 \text{ then } \Sigma_s \text{ is the "isomorphic" shear stress, } \sigma_s = \left\| \underset{\approx}{S} \right\| \tag{204}$$

Typical/convenient choices for the isotropic stress coefficient are

$$\text{if } \gamma_p = -\tfrac{1}{\sqrt{3}} \text{ then } \Sigma_p \text{ is the conventional pressure } p \text{ (positive in compression)} \tag{205}$$

$$\text{if } \gamma_p = \tfrac{1}{\sqrt{3}} \text{ then } \Sigma_p \text{ is the tensile pressure (positive in tension)} \tag{206}$$

$$\text{if } \gamma_p = 1 \text{ then } \Sigma_p \text{ is the "isomorphic" pressure, } \sigma_p = \underset{\approx}{\sigma} : \hat{\underset{\approx}{I}} = \frac{1}{\sqrt{3}} \mathrm{tr}(\underset{\approx}{\sigma}) \tag{207}$$

Recall that the stress tensor can be written

$$\underset{\approx}{\sigma} = \sigma_p \hat{\underset{\approx}{I}} + \sigma_s \hat{\underset{\approx}{S}} \tag{208}$$

In this form, $\hat{\underset{\approx}{I}}$ and $\hat{\underset{\approx}{S}}$ are like an orthonormal basis for the hyperplane containing $\underset{\approx}{\sigma}$ and $\underset{\approx}{I}$. Using the stress measures of Eqs. (201) and (200), the stress can be written

$$\underset{\approx}{\sigma} = \frac{\Sigma_p}{\gamma_p} \hat{\underset{\approx}{I}} + \frac{\Sigma_s}{\gamma_s} \hat{\underset{\approx}{S}} \tag{209}$$

In this form, $\{\Sigma_p, \Sigma_s\}$ can be regarded as coefficients of the stress tensor with respect to the orthogonal *but not normalized* basis $\hat{\underset{\approx}{I}}/\gamma_p$ and $\hat{\underset{\approx}{S}}/\gamma_s$. There is nothing wrong with this *per se*. However, as discussed below, engineering stress measures for which $\gamma_p \neq 1$ and/or $\gamma_s \neq 1$ basically distort your 2D visualizations of stress space — visualizations plotted as Σ_s vs. Σ_p will not be representative of what's really happening in 6-D stress space. For example, if the yield surface is a perfect *sphere* in 6-D stress space, then it will be an *ellipse* in the Σ_s vs. Σ_p plane if $\gamma_p \neq \gamma_s$. The 6D spherical yield surface would at least be a circle in the Σ_s vs. Σ_p plane if γ_p and γ_s are equal, but it will have not have

the same radius as the 6D sphere unless $\gamma_p = \gamma_s = 1$. The only way to accurately depict stress space in a 2D plot is to use the isomorphic stress measures corresponding to $\gamma_p = \gamma_s = 1$. Any other values of γ_p and γ_s result in rather ugly corrections in the key equations, as we shall demonstrate below.

Recall that the yield function $f(\underset{\sim}{\sigma})$ generally depends on the full stress tensor. Pressure-dependent Von Mises yield models are greatly simplified because the yield function depends only on the pressure and the magnitude of the stress deviator. In other words, there exists a function $g(\Sigma_s, \Sigma_p)$ such that

$$f(\underset{\sim}{\sigma}) = g(\Sigma_s, \Sigma_p) \tag{210}$$

In the pie-plane*, the yield surface is a circle whose radius depends on the pressure. The expression $g(\Sigma_s, \Sigma_p) = 0$ implicitly defines a yield curve in which the deviatoric stress measure Σ_s is a function of isotropic stress measure Σ_p. In an *explicit* relationship, $\Sigma_s = G(\Sigma_p)$, is known then a corresponding implicit function is constructed by $g(\Sigma_s, \Sigma_p) = \Sigma_s - G(\Sigma_p)$.

A natural way to visualize the yield surface is to show the yield curve in the "Rendulic" plane where Σ_s is plotted against Σ_p. When doing this, however, it is important to realize that *the yield surface in the Rendulic (Σ_s vs. Σ_p) plane will not have the same size and shape as the yield surface in stress space unless isomorphic stress measures are used*. The yield surface in the Rendulic plane will have the same shape as the yield surface in stress space if and only if $\gamma_p = \gamma_s$. It will have the same size only if $\gamma_p = \gamma_s = 1$. As shown in Fig. 8, if the analyst chooses $\gamma_p \neq \gamma_s$, then *the Rendulic yield surface will be a distorted representation of the true yield surface, and vectors that were normal to the isomorphic yield surface will not be normal to the yield surface in the engineering stress plane!* This is why we highly recommend using isomorphic stress measures to visualize the yield curve.

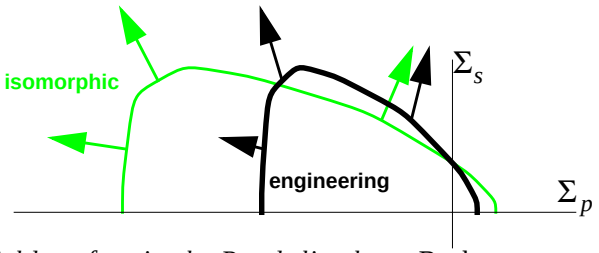

Figure 8. The yield surface in the Rendulic plane. Both curves are plots of $g(\Sigma_s, \Sigma_p) = 0$. The larger yield surface corresponds to isomorphic stress measures and would be the most geometrically accurate representation of the actual yield surface in stress space. The smaller yield surface corresponds to using the tensile pressure and the magnitude of the stress deviator as the stress measures. This choice distorts the yield surface so that the "normals" are no longer perpendicular to the yield surface.

*so named because it looks like a piece of pie. It is often written and π-plane or pi-plane.

To apply return methods to a pressure-dependent Von Mises yield surface, we need the gradient of the yield function $f(\underset{\approx}{\sigma})$ with respect to stress. Applying the chain rule to Eq. (210) gives

$$\underset{\approx}{B} = \frac{df(\underset{\approx}{\sigma})}{d\underset{\approx}{\sigma}} = g_{,s}\frac{d\Sigma_s}{d\underset{\approx}{\sigma}} + g_{,p}\frac{d\Sigma_s}{d\underset{\approx}{\sigma}}, \tag{211}$$

where

$$g_{,s} \equiv \frac{\partial g(\Sigma_s, \Sigma_p)}{\partial \Sigma_s} \quad \text{and} \tag{212a}$$

$$g_{,p} \equiv \frac{\partial g(\Sigma_s, \Sigma_p)}{\partial \Sigma_p} \tag{212b}$$

Using Eq. (199) in Eqs. (201) and (200) shows that

$$\frac{d\Sigma_s}{d\underset{\approx}{\sigma}} = \gamma_s \hat{\underset{\approx}{S}} \quad \text{and} \tag{213a}$$

$$\frac{d\Sigma_p}{d\underset{\approx}{\sigma}} = \gamma_p \hat{\underset{\approx}{I}} \tag{213b}$$

Therefore, the normal to the yield surface in stress space is

$$\boxed{\underset{\approx}{B} = \gamma_p\, g_{,p}\hat{\underset{\approx}{I}} + \gamma_s\, g_{,s}\hat{\underset{\approx}{S}}} \tag{214}$$

When viewed as a vector in stress space, we note that $\underset{\approx}{B}$ is contained in the hyperplane formed by the unit tensors $\hat{\underset{\approx}{I}}$ and $\hat{\underset{\approx}{S}}$. The components of $\underset{\approx}{B}$ in that stress hyperplane are $\{(\gamma_p g_{,p}), (\gamma_s g_{,s})\}$. Note that the normal to the yield surface in the Rendulic plane is a two-dimensional vector with components $\{g_{,p}, g_{,s}\}$. Hence, the normal to the Rendulic yield surface is isomorphic to the normal in stress space only if $\gamma_p = \gamma_s = 1$. This is a principal argument in favor of using the isomorphic stress measures. We are presenting the analysis for general values of γ_p and γ_s only so that existing models that use more conventional stress measures can be easily implemented.

The radial return algorithm requires a return direction tensor. Recall that the return direction in stress space must be parallel to

$$\underset{\approx}{A} \equiv \underset{\approx\approx\approx\approx}{E} : \underset{\approx}{M}. \tag{215}$$

where $\underset{\approx}{M}$ is the (known) direction of the plastic strain rate. For the following discussion, we presume that $\underset{\approx}{M}$ has no component perpendicular to the Rendulic plane; in other words, we assume that $\underset{\approx}{M}$ can be expressed as a linear combination of $\hat{\underset{\approx}{I}}$ and $\hat{\underset{\approx}{S}}$:

$$\underset{\approx}{M} = M_p \hat{\underset{\approx}{I}} + M_s \hat{\underset{\approx}{S}} \tag{216}$$

If the elastic stiffness is isotropic, Eq. (215) becomes

$$\underset{\approx}{A} \equiv \underset{\approx\approx}{E} : \underset{\approx}{M} = (3K M_p)\hat{\underset{\approx}{I}} + (2G M_s)\hat{\underset{\approx}{S}} \tag{217}$$

where K is the bulk modulus and G is the shear modulus. Factoring out the bulk modulus, the return direction can be written

$$\underset{\approx}{A} = 3K[\, M_p \hat{\underset{\approx}{I}} + \eta\, M_s \hat{\underset{\approx}{S}}\,] \tag{218}$$

where

$$\boxed{\eta = \frac{2G}{3K} = \frac{1-2\nu}{1+\nu}} \tag{219}$$

Here, ν is Poisson's ratio. Recall that only the *direction* of $\underset{\approx}{A}$ matters — any scalar multiple of the projection direction is inconsequential. Thus, the factor $3K$ in Eq. (218) has no influence, and the projection direction may be replaced without loss in generality by

$$\boxed{\underset{\approx}{A} = M_p \hat{\underset{\approx}{I}} + \eta\, M_s \hat{\underset{\approx}{S}}} \tag{220}$$

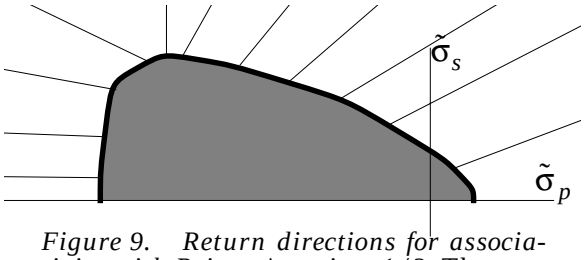

Figure 9. Return directions for associativity with Poisson's ratio = 1/3. The return direction is oblique to the yield surface even though the plastic strain rate is normal to the yield surface!

Comparing Eqs. (216) and (220) shows that the projection direction will not be aligned with the plastic strain rate except in the exceptional case of ν=0. Excluding negative Poisson's ratios, η will lie in the range $(0, 1]$, so the projection direction will generally have a shallower slope in the Rendulic plane than the plastic strain rate. If, for example, Poisson's ratio equals 1/3, then $\eta = 1/4$ and (when drawn in the isomorphic Rendulic plane) the slope of the return direction will be 1/4 that of the plastic strain rate. For an associative flow rule, the plastic strain rate is normal to the yield surface, and the above result shows that the return direction must then be *oblique* to the yield surface, as sketched in Fig. 9.

Though often applied without physical justification, modern plasticity codes often use a "Prandtl" return direction to project the trial stress back to the yield surface. With this approach, the magnitude of the stress deviator is reduced enough to place the stress on the yield surface. Hence, the Prandtl rule corresponds to a projection direction $\underset{\approx}{A}$ that is aligned with $\underset{\approx}{S}$. Referring to Eq. (216), this implies that $M_p = 0$. In other words, the Prandtl return direction corresponds to a purely deviatoric plastic strain rate. Hence, for isotropic materials with a Von Mises yield criterion, a Prandtl return direction

holds if and only if the material plastically incompressible. The Prandtl return direction seems nominally applicable to materials having the properties that (1) the elastic response is isotropic, (2) the yield stress varies with pressure and (3) the material is plastically incompressible. A Prandtl return direction would be highly inappropriate for porous metals or any other material that exhibits non-negligible amounts of irreversible permanent volume change.

These preliminary discussions have revealed how complicated and subtle pressure-sensitivity can be. We now proceed to an algorithm for returning the stress to the yield surface along the proper return direction for any plastic strain rate direction. Such algorithms typically require the stress at the beginning of the step and the total rate (assumed constant throughout the step). The algorithm computes the trial elastic stress and then returns it to the yield surface to obtain the final updated stress at the end of the step. The algorithm also outputs the plastic strain rate and updates the scalar measures of the plastic strain rate. The "plastic segment" is a scalar measure of the total accumulated plastic strain and is defined to be the integral over time of the magnitude of the plastic strain rate tensor. Thus, since the plastic strain rate direction $\underset{\sim}{M}$ is a unit tensor, the plastic segment is defined:

$$\lambda = \int_0^t \dot{\lambda}\, dt = \int_0^t \left\| \dot{\underset{\sim}{\varepsilon}}^p \right\| dt \tag{221}$$

Recall that a principal advantage of return algorithms is that they do not require an explicit determination of what fraction of the time step is plastic. However, computing the increment in λ can be tricky $\dot{\lambda}$ is zero during elastic intervals. Referring the governing equations on page 10, we note that the only equation that holds over both intervals is $\dot{\underset{\sim}{\sigma}} = \underset{\approx}{E} : \dot{\underset{\sim}{\varepsilon}}^e$. Therefore, once the stress is updated via return methods, the total stress rate can be computed via standard finite difference and the plastic strain rate tensor may be approximated by

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^e = \dot{\underset{\sim}{\varepsilon}} - \underset{\approx}{E}^{-1} : \dot{\underset{\sim}{\sigma}}. \tag{222}$$

This formula may be substituted into Eq. (221) to update the plastic segment. This formula may also be used to compute another popular scalar measure of plastic strain is the so-called equivalent plastic strain, defined

$$\varepsilon_p = \int_0^t \sqrt{\dot{\underset{\sim}{\varepsilon}}^{p\prime} : \dot{\underset{\sim}{\varepsilon}}^{p\prime}}\, dt \text{, where } \dot{\underset{\sim}{\varepsilon}}^{p\prime} \text{ is the deviatoric part of } \dot{\underset{\sim}{\varepsilon}}^p. \tag{223}$$

Solution algorithm for arbitrary plastic strain directions. In this section, we provide a general procedure for projecting the stress back to the yield surface for isotropic elasticity a pressure-dependent Von Mises yield surface with a (permissibly) nonassociative flow rule. The method presented here includes crude corrections for yield surface curvature.

INPUT:

- γ_p, the constant from Eq. (200) for the preferred isotropic stress measure.
- γ_s, the constant from Eq. (201) for the preferred deviatoric stress measure.
- Δt, the time step
- $\dot{\underset{\approx}{\varepsilon}}'$, the deviatoric part of the total strain rate tensor.
- $\dot{\varepsilon}_v$, the trace of the total strain rate tensor
- K, the elastic tangent bulk modulus
- G, the elastic tangent shear modulus
- λ^{old}, the old value of the plastic segment.
- $\varepsilon_p^{\text{old}}$, the equivalent plastic shear strain at the beginning of the step.
- A routine that will evaluate the yield function $g(\sigma_s, \sigma_p)$
- Routines that will evaluate the yield function derivatives
  $g_{,p} \equiv \dfrac{\partial g(\sigma_s, \sigma_p)}{\partial \sigma_p}$ and $g_{,s} \equiv \dfrac{\partial g(\sigma_s, \sigma_p)}{\partial \sigma_s}$
- A routine that will provide the unit tensor $\underset{\approx}{M}$ in the direction of the plastic strain rate. This is not needed if an associative flow rule is desired.
- $\underset{\approx}{S}^{\text{old}}$, stress deviator at the beginning of the time step.
- p^{old}, conventional pressure $(-\frac{1}{3}\text{tr}\underset{\approx}{\sigma})$ at the beginning of the step.

OUTPUT:

- $\underset{\approx}{S}^{\text{new}}$, stress deviator at the end of the time step.
- p^{new}, conventional pressure $(-\frac{1}{3}\text{tr}\underset{\approx}{\sigma})$ at the end of the step.

λ^{new}, the value of the plastic segment. at the end of the step

$\varepsilon_p^{\text{new}}$, the equivalent plastic strain at the end of the step.

$\dot{\underset{\approx}{\varepsilon}}^p$, the plastic strain rate.

ALGORITHM:

STEP 1.Apply Eq. (200) to compute $\sigma_p^{\text{old}} = -\sqrt{3}\gamma_p p^{\text{old}}$.

STEP 2.Apply Eq. (201) to compute $\sigma_s^{\text{old}} = \gamma_s \left\| \underset{\approx}{S}^{\text{old}} \right\|$.

STEP 3.Compute the trial stress deviator:

$$\underset{\approx}{S}^{\text{trial}} = \underset{\approx}{S}^{\text{old}} + 2G(\dot{\underset{\approx}{\varepsilon}}' \Delta t) \tag{224}$$

STEP 4.Compute the trial pressure:

$$p^{\text{trial}} = p^{\text{old}} - K\dot{\varepsilon}_v \Delta t \tag{225}$$

STEP 5.Apply Eq. (200) to compute $\sigma_p^{\text{trial}} = -\sqrt{3}\gamma_p p^{\text{trial}}$.

STEP 6.Compute $\left\| \underset{\approx}{S}^{\text{trial}} \right\| = \sqrt{\underset{\approx}{S}^{\text{trial}} : \underset{\approx}{S}^{\text{trial}}}$.

STEP 7.Apply Eq. (201) to compute $\sigma_s^{\text{trial}} = \gamma_s \left\| \underset{\approx}{S}^{\text{trial}} \right\|$.

STEP 8.Call the yield function to compute $g^{\text{trial}} = g(\sigma_s^{\text{trial}}, \sigma_p^{\text{trial}})$.

STEP 9.If $g^{\text{trial}} > 0$, the step is at least partly plastic, so proceed to STEP 10.
If $g^{\text{trial}} \le 0$ then the step is elastic, so do the following:

(i) Set $\dot{\underset{\approx}{\varepsilon}}^p = \underset{\approx}{0}$

(ii) Set $\underset{\approx}{S}^{\text{new}} = \underset{\approx}{S}^{\text{trial}}$

(iii) Set $p^{\text{new}} = p^{\text{trial}}$

(iv) Set $\lambda^{\text{new}} = \lambda^{\text{old}}$

(v) Set $\varepsilon_p^{\text{new}} = \varepsilon_p^{\text{old}}$

(vi) Go to step 22.

STEP 10.Compute $\hat{\underset{\approx}{S}} = \dfrac{\underset{\approx}{S}^{\text{trial}}}{\left\| \underset{\approx}{S}^{\text{trial}} \right\|}$. This will be taken as the final stress deviator direction. The remainder of this algorithm applies oblique return methods in the Rendulic plane to put the stress back on the yield surface.

STEP 11.Set $g^{\text{test}} = g^{\text{trial}}$.
Set $\sigma_p^{\text{test}} = \sigma_p^{\text{trial}}$.
Set $\sigma_s^{\text{test}} = \sigma_s^{\text{trial}}$.

STEP 12.Compute the yield surface derivatives at the test stress

$$g_{,p}^{\text{test}} = g_{,p}(\sigma_p^{\text{test}}, \sigma_s^{\text{test}})$$
$$g_{,s}^{\text{test}} = g_{,s}(\sigma_p^{\text{test}}, \sigma_s^{\text{test}})$$

STEP 13. Approximate the isomorphic components of $\underset{\approx}{B}$ by applying Eq. (214) at the test stress. Specifically,

Set $\tilde{B}_p = \gamma_p g^{\text{test}}_{,p}$ and

Set $\tilde{B}_s = \gamma_s g^{\text{test}}_{,s}$.

STEP 14. For an associative flow rule, set $\tilde{M}_p = \tilde{B}_p$ and $\tilde{M}_s = \tilde{B}_s$. For a nonassociative flow rule, call the appropriate routine to compute the direction of the plastic strain rate. In other words, compute $\tilde{M}_p$ and $\tilde{M}_s$ so that $\underset{\approx}{M} = \tilde{M}_p \hat{\underset{\approx}{I}} + \tilde{M}_s \hat{\underset{\approx}{S}}$. In this algorithm, it is not necessary for you to force $\underset{\approx}{M}$ to be a unit tensor. It is also not necessary to store the actual tensor for $\underset{\approx}{M}$ — only the Rendulic components are needed.

STEP 15. Compute $\eta = \dfrac{2G}{3K}$

STEP 16. Apply Eq. (220) to obtain the isomorphic components of the projection direction: $\tilde{A}_p = \tilde{M}_p$ and $\tilde{A}_s = \eta \tilde{M}_s$. Mathematically, the projection direction can be harmlessly multiplied by any scalar without affecting the solution. For numerical reasons, the components of the projection direction should be scaled so that they are on the order of stress. One way to do this would be to multiply both projection direction components by

$$\sqrt{\frac{(\sigma_p^{\text{test}})^2 + (\sigma_s^{\text{test}})^2}{(\tilde{A}_p)^2 + (\tilde{A}_s)^2}}$$

STEP 17. We know that the desired stress is a projection of the test stress back to the yield surface. Thus we know that

$\tilde{\sigma}_p = \tilde{\sigma}_p^{\text{test}} + \beta \tilde{A}_p$ and

$\tilde{\sigma}_s = \tilde{\sigma}_s^{\text{test}} + \beta \tilde{A}_s$

Written in terms of the analyst's preferred stress measures, we know that

$\sigma_p = \sigma_p^{\text{test}} + \beta \gamma_p \tilde{A}_p$ and

$\sigma_s = \sigma_s^{\text{test}} + \beta \gamma_s \tilde{A}_s$

We seek the value of β such that $g(\sigma_p(\beta), \sigma_s(\beta)) = 0$. We shall iteratively solve this using a slight variation on Newton's method that is expected to give better results near regions of high curvature on the yield surface. Apply Newton's method starting with $\beta = 0$, to compute an improved estimate:

$$\beta^{\text{next}} = -\frac{g^{\text{test}}}{g^{\text{test}}_{,p}\gamma_p \tilde{A}_p + g^{\text{test}}_{,s}\gamma_s \tilde{A}_s}$$

One advantage of this formulation is that no additional function calls are needed to improve the estimate for β.

STEP 18. Compute the improved estimates for the test stress by replacing

$$\sigma_p^{\text{test}} := \sigma_p^{\text{test}} + \beta^{\text{next}}\gamma_p \tilde{A}_p$$

$$\sigma_s^{\text{test}} := \sigma_s^{\text{test}} + \beta^{\text{next}}\gamma_s \tilde{A}_s$$

STEP 19. The projection direction has already been scaled to be on the order of stress. Test for convergence by checking whether β^{next} is sufficiently tiny in comparison to one. If so, go to STEP 20. If not, compute $g^{\text{test}} = g(\sigma_s^{\text{trial}}, \sigma_p^{\text{trial}})$ and loop back to STEP 12.

STEP 20. At this point, the stress measures have been updated. Compute the final updated pressure and stress deviator by

$$p^{\text{new}} = \frac{\sigma_p^{\text{test}}}{-\sqrt{3}\gamma_p}$$

$$\underset{\approx}{S}^{\text{new}} = \frac{\sigma_s^{\text{test}}}{\gamma_s}\hat{\underset{\approx}{S}}$$

STEP 21. Now finish up by computing the other promised outputs. Recall that $\dot{\underset{\approx}{\varepsilon}}^e = \underset{\approx\approx}{E}^{-1} : \dot{\underset{\approx}{\sigma}}$ and $\dot{\underset{\approx}{\varepsilon}}^p = \dot{\underset{\approx}{\varepsilon}} - \dot{\underset{\approx}{\varepsilon}}^e$. The deviatoric plastic strain rate is

$$\dot{\underset{\approx}{\varepsilon}}^{p\prime} = \dot{\underset{\approx}{\varepsilon}}' - \frac{1}{2G}\left(\frac{\underset{\approx}{S}^{\text{new}} - \underset{\approx}{S}^{\text{old}}}{\Delta t}\right)$$

Compute the volumetric plastic strain rate by $\dot{\underset{\approx}{\varepsilon}}^p = \dot{\underset{\approx}{\varepsilon}} - \dot{\underset{\approx}{\varepsilon}}^e$

$$\dot{\varepsilon}_v^p = \dot{\varepsilon}_v - \frac{1}{K}\left(\frac{p^{\text{new}} - p^{\text{old}}}{\Delta t}\right) = \frac{1}{K}\left(\frac{p^{\text{trial}} - p^{\text{new}}}{\Delta t}\right)$$

Update the distortional plastic strain by

$$\varepsilon_p^{\text{new}} = \varepsilon_p^{\text{old}} + \sqrt{\dot{\underset{\approx}{\varepsilon}}^{p\prime} : \dot{\underset{\approx}{\varepsilon}}^{p\prime}}$$

Update the total equivalent plastic strain by

$$\lambda^{\text{new}} = \lambda^{\text{old}} + \sqrt{(\dot{\underset{\approx}{\varepsilon}}^{p\prime} : \dot{\underset{\approx}{\varepsilon}}^{p\prime}) + \tfrac{1}{3}\dot{\varepsilon}_v^2}\,.$$

STEP 22. Stop.

WORKSHOP:
Geometrical interpretation of Radial and Oblique Return Methods

PLASTICITY 2000:
"The Eighth International Symposium on Plasticity
and its Current Applications"
Session FA I-1: Computational Plasticity and Viscoplasticity
Whistler, CANADA
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ABSTRACT: Return algorithms are probably the most popular means of numerically solving conventional plasticity equations. The basic tenets of these techniques are here rigorously justified and interpreted geometrically in 6D stress space. For any return algorithm, the first step is to tentatively assume elastic behavior throughout a given time step. If the resulting "trial" stress is forbidden (i.e. if it violates the yield condition), then the tentative assumption of elastic response is rejected. Even when it is found to violate the yield condition, the trial stress is nevertheless useful because it can then be projected back to the plastic yield surface to give the updated stress. The return algorithm is called "normal" or "orthogonal" if the trial stress is projected directly to the nearest point on the yield surface. The return method is called "radial" or "Prandtl" when the projection is accomplished by reducing the magnitude of the trial stress deviator. Return algorithms are often wrongly regarded as numerical "tricks" because they appear to be ad hoc means of keeping the stress on the yield surface. It is natural to inquire whether other approaches might be more accurate for the same computational cost, but it is shown here that return methods are rigorously justifiable and appear to correspond to optimal numerical accuracy and efficiency. It is shown that issues such as plastic stability, dissipation, and convexity dictate appropriate choices for the quantities that are presumed known in the derivation of return algorithms; it is not the return algorithm per se that addresses such physical concerns. It is proved that the correct return direction is dictated by the governing equations and is not aligned with the plastic strain rate except under certain conditions. Consequently, normality of the plastic strain rate does not necessarily correspond to normality of the return direction, and vice versa. These claims are proved first in the context of stationary yield surfaces and then generalized to permit hardening or softening. The technical note is intended to provide nothing more than geometrical insight into known results.

Tensors are vectors!

To a mathematician, a vector is a member of a set for which *addition* and *scalar multiplication* satisfy certain rules.

Many familiar 3D vector concepts and theorems also apply to tensors when regarded as 9D vectors.

3D vector operations

$\underset{\sim}{u} = \alpha \underset{\sim}{c}$ *means* $u_i = \alpha c_i$

$\underset{\sim}{v} = \underset{\sim}{a} + \underset{\sim}{b}$ *means* $v_i = a_i + b_i$

9D tensor operations

$\underset{\approx}{U} = \alpha \underset{\approx}{C}$ *means* $U_{ij} = \alpha C_{ij}$

$\underset{\approx}{V} = \underset{\approx}{A} + \underset{\approx}{B}$ *means* $V_{ij} = A_{ij} + B_{ij}$

3D inner product

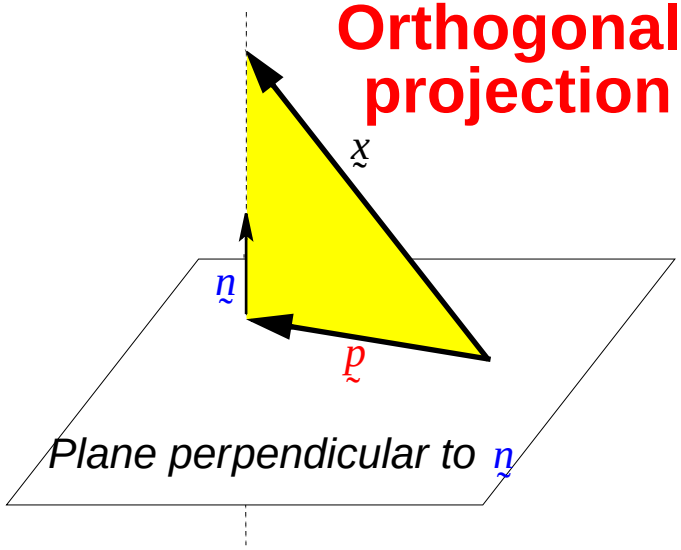
$$\underset{\sim}{r} \bullet \underset{\sim}{s} \text{ means } \sum_{i=1}^{3} r_i s_i$$

9D inner product

$$\underset{\approx}{R} : \underset{\approx}{S} \text{ means } \sum_{i=1}^{3} \sum_{j=1}^{3} R_{ij} S_{ij}$$

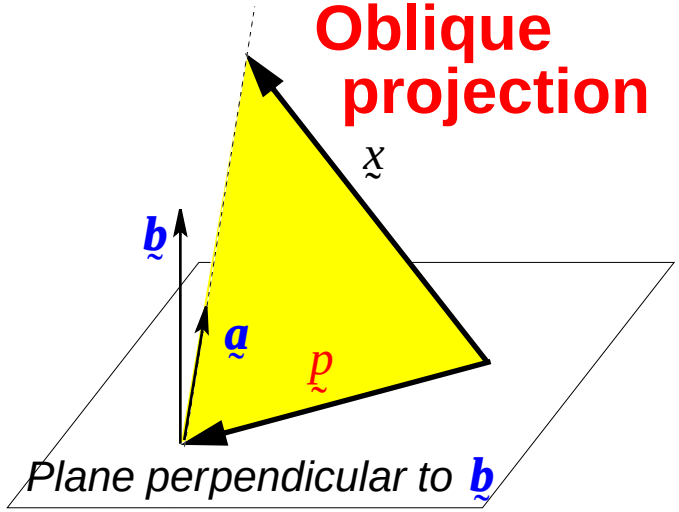
Projection operations

Orthogonal projection



$$\underset{\sim}{p} = \underset{\sim}{x} - \underset{\sim}{n}(\underset{\sim}{n} \bullet \underset{\sim}{x})$$

Oblique projection



$$\underset{\sim}{p} = \underset{\sim}{x} - \frac{\underset{\sim}{a}(\underset{\sim}{b} \bullet \underset{\sim}{x})}{\underset{\sim}{a} \bullet \underset{\sim}{b}}$$

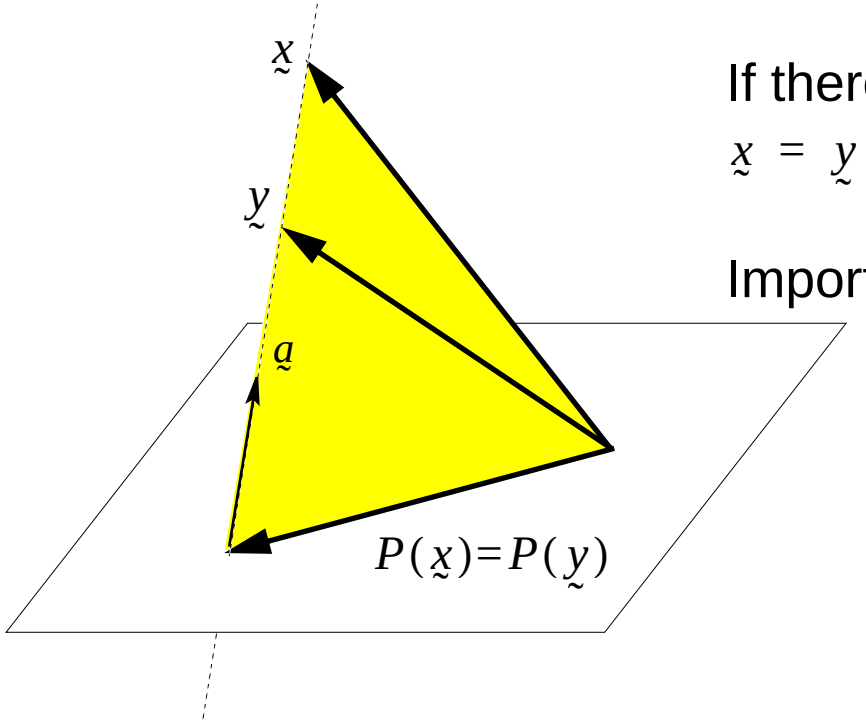
Note: $\underset{\sim}{b}$ defines the target plane; $\underset{\sim}{a}$ defines projection direction.

Analog for 9D tensor space:

$$P(\underset{\approx}{X}) = \underset{\approx}{X} - \frac{\underset{\approx}{A}(\underset{\approx}{B} : \underset{\approx}{X})}{\underset{\approx}{A} : \underset{\approx}{B}}$$

Projections are linear. . . $P(\alpha_1 \underset{\approx}{X}_1 + \alpha_2 \underset{\approx}{X}_2) = \alpha_1 P(\underset{\approx}{X}_1) + \alpha_2 P(\underset{\approx}{X}_2)$

LEMMA



If there is a β such that
$\underset{\sim}{x} = \underset{\sim}{y} + \beta \underset{\sim}{a}$, then $P(\underset{\sim}{x})=P(\underset{\sim}{y})$.

Important: *converse is true too!*

Analog for tensors:

If $\underset{\approx}{X} = \underset{\approx}{Y} + \beta \underset{\approx}{A}$ then $P(\underset{\approx}{X})=P(\underset{\approx}{Y})$ and vice versa.

Corollary: $P(P(\underset{\approx}{X})) = P(\underset{\approx}{X})$ (projecting twice makes no change).

Nonhardening plasticity

Known:

$\underset{\approx}{B}$, gradient of yield function ($B_{ij} = \partial f / \partial \sigma_{ij}$).

$\dot{\underset{\approx}{\varepsilon}}$, total strain rate.

$\underset{\approx\approx}{E}$, fourth-order elastic tangent stiffness tensor.

$\underset{\approx}{M}$, direction of the plastic strain rate.

Unknown:

$\dot{\underset{\approx}{\sigma}}$, rate of stress

$\dot{\underset{\approx}{\varepsilon}}^e$, elastic part of the strain rate

$\dot{\underset{\approx}{\varepsilon}}^p$, plastic part of the strain rate.

$\dot{\lambda}$, magnitude of the plastic part of the strain rate.

Governing equations

$\dot{\underset{\approx}{\varepsilon}} = \dot{\underset{\approx}{\varepsilon}}^e + \dot{\underset{\approx}{\varepsilon}}^p$ *strain rate decomposition*

$\dot{\underset{\approx}{\varepsilon}}^p = \dot{\lambda}\underset{\approx}{M}$ *plastic strain direction is known*

$\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}}^e$ *stress linear in elastic strain*

$\underset{\approx}{B} : \dot{\underset{\approx}{\sigma}} = 0$ *stress stays on yield surface*

Solution:

Note $\dot{\underset{\approx}{\varepsilon}}^e = \dot{\underset{\approx}{\varepsilon}} - \dot{\underset{\approx}{\varepsilon}}^p = \dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M}$, so $\dot{\underset{\approx}{\sigma}} = \underset{\approx\approx}{E} : (\dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M})$. For convenience,

define $\dot{\underset{\approx}{\sigma}}^{\text{trial}} = \underset{\approx\approx}{E} : \dot{\underset{\approx}{\varepsilon}}$ and $\underset{\approx}{A} = \underset{\approx\approx}{E} : \underset{\approx}{M}$. Then $\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \dot{\lambda}\underset{\approx}{A}$

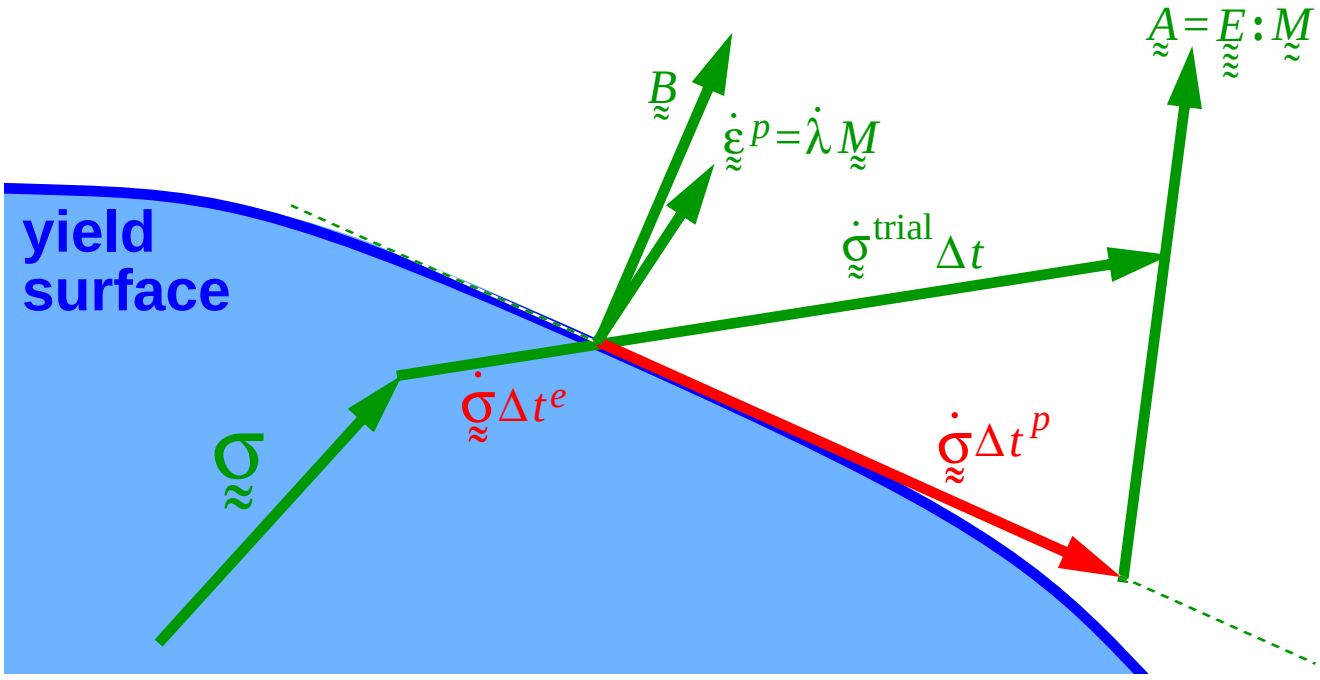
Enforce last equation to get $\underset{\approx}{B} : [\dot{\underset{\approx}{\sigma}}^{\text{trial}} - \dot{\lambda}\underset{\approx}{A}] = 0$. Solve for $\dot{\lambda}$ and back

substitute to get solution for stress rate: $\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \left(\frac{\underset{\approx}{B} : \dot{\underset{\approx}{\sigma}}^{\text{trial}}}{\underset{\approx}{B} : \underset{\approx}{A}} \right) \underset{\approx}{A}$.

Geometrical interpretation

Slightly rearrange solution to final form:

$$\dot{\underset{\approx}{\sigma}} = P(\dot{\underset{\approx}{\sigma}}^{\text{trial}}) \qquad where \qquad P(\underset{\approx}{X}) = \underset{\approx}{X} - \frac{\underset{\approx}{A}(\underset{\approx}{B}:\underset{\approx}{X})}{\underset{\approx}{A}:\underset{\approx}{B}}$$



Numerical solution: $\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^{\text{trial}} + \beta\underset{\approx}{A}$. Find β by $f(\underset{\approx}{\sigma}^{\text{trial}} + \beta\underset{\approx}{A}) = 0$.

http://www.me.unm.edu/~rmbrann/gobag.html

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/home/rmbrann/Teach/MtlModels/RadialReturn/plas2000vug

Discussion

The return direction is...

- coaxial with $\underset{\approx}{A}$.
- not generally normal to the yield surface.
- not generally aligned with the plastic strain rate.
- not dictated by physical considerations such as positive dissipation, yield surface convexity, or plastic stability. (Such concerns dictate appropriate values for “known” quantities.)
- “radial” if and only if the material is plastically incompressible.

The above analysis can be generalized (see web document) to include hardening/softening. Projection of the trial stress back to the *current* yield surface remains valid even though the stress rate is no longer a projection of the trial stress rate.

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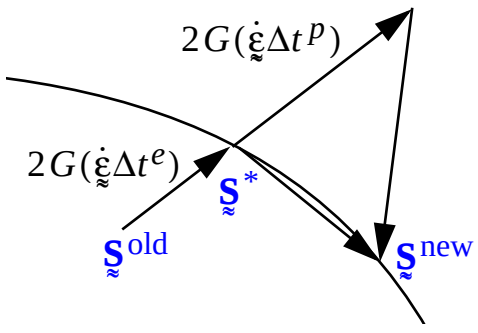
Equivalent plastic strain

Many constitutive models use yield surface evolution laws that depend on the so-called "equivalent plastic strain," which is defined

$$\gamma_p \equiv \int \sqrt{\frac{2}{3}\dot{\underset{\approx}{\varepsilon}}^{p\prime} : \dot{\underset{\approx}{\varepsilon}}^{p\prime}}\, dt = \sqrt{\frac{2}{3}} \int \left\| \dot{\underset{\approx}{\varepsilon}}^{p\prime} \right\| dt$$

The best method uses the definition directly:

$$\Delta\gamma_p \equiv \sqrt{\frac{2}{3}} \left\| \dot{\underset{\approx}{\varepsilon}}' - \dot{\underset{\approx}{\varepsilon}}^{e\prime} \right\| \Delta t \text{, or, for isotropic, } \Delta\gamma_p \equiv \sqrt{\frac{2}{3}} \left\| \dot{\underset{\approx}{\varepsilon}}' - \frac{\dot{\underset{\approx}{\mathbf{S}}}}{2G} \right\| \Delta t .$$



For a finite time step Δt,

$$\gamma_p^{\text{new}} \equiv \gamma_p^{\text{old}} + \sqrt{\frac{2}{3}} \left\| \dot{\underset{\approx}{\varepsilon}}' \Delta t - \frac{\underset{\approx}{\mathbf{S}}^{\text{new}} - \underset{\approx}{\mathbf{S}}^{\text{old}}}{2G} \right\|$$

(...better suited for partially plastic intervals.)

Supplemental topic: invariant yield functions

Tresca: Stress state is below yield *if and only if*

$$f(\underset{\approx}{\sigma}) = \frac{1}{2}\max(|\sigma_1-\sigma_2|, |\sigma_2-\sigma_3|, |\sigma_3-\sigma_1|) - k < 0 \qquad (1)$$

Some authors (e.g. Fung, 1965, Lubliner 1990) **wrongly** claim that an acceptable *alternative* Tresca yield function is

$$F(\underset{\approx}{\sigma}) = [(\sigma_1-\sigma_2)^2-4k^2][(\sigma_2-\sigma_3)^2-4k^2][(\sigma_3-\sigma_1)^2-4k^2]. \qquad (2)$$

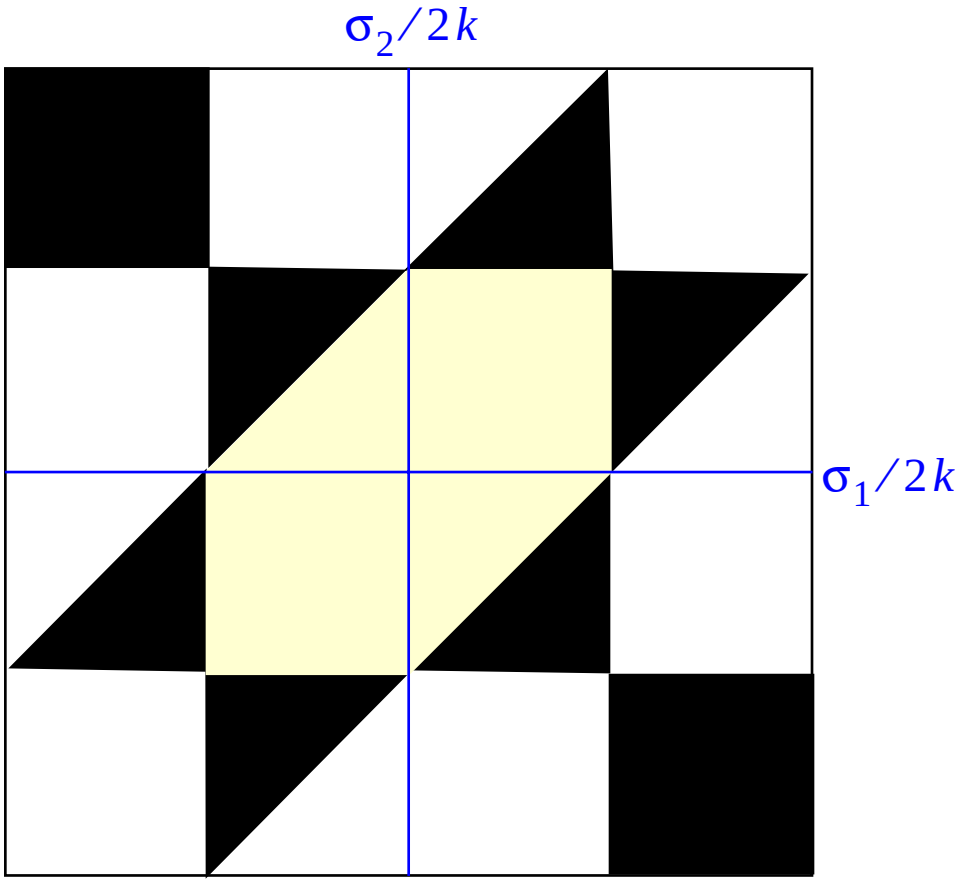
This is intoxicating because it can be written with invariants as

$$F(\underset{\approx}{\sigma}) = 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 \qquad (3)$$

FATAL FLAW: If stress is below yield, then $F(\underset{\approx}{\sigma}) \le 0$, but converse is false! A return algorithm using F might wrongly think a plastic trial stress is below yield. For example, $\sigma_1 = \sigma_2 = 3k$ and $\sigma_3 = 0$ is correctly identified to be above yield by $f(\underset{\approx}{\sigma})$, but not by $F(\underset{\approx}{\sigma})$.

Plot of (bad) invariant Tresca function

Under the assumption of plane stress where $\sigma_3 = 0$, regions where $F(\underset{\approx}{\sigma}) > 0$ are shown in black. *A valid yield function should be black everywhere outside the yellow Tresca hexagon*.



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Supplemental topic: 9D vector basis

Recall that tensors are 9D vectors, so we may define a 9×1 component array for them: $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9 = \{T_{11}, T_{21}, T_{31}, T_{12}, T_{22}, T_{32}, T_{13}, T_{23}, T_{33}\}$,

3D vector basis expansion

$$\underset{\sim}{v} = v_1 \underset{\sim}{\mathbf{e}}_1 + v_2 \underset{\sim}{\mathbf{e}}_2 + v_3 \underset{\sim}{\mathbf{e}}_3$$

Summation form

$$\underset{\sim}{v} = \sum_{k=1}^{3} v_k \underset{\sim}{\mathbf{e}}_k$$

9D tensor expansion

$$\begin{aligned}\underset{\approx}{T} = {} & T_{11}\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1 + T_{12}\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2 + T_{13}\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3 \\ & + T_{21}\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1 + T_{22}\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2 + T_{23}\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 \\ & + T_{31}\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + T_{32}\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2 + T_{33}\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3\end{aligned}$$

Summation form

$$\underset{\approx}{T} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\underset{\sim}{\mathbf{e}}_i\underset{\sim}{\mathbf{e}}_j = \sum_{K=1}^{9} T^o_K \underset{\approx}{\xi}^o_K$$

where $T^o_1 = T_{11}$, $\underset{\approx}{\xi}^o_1 = \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1$, $T^o_2 = T_{21}$, $\underset{\approx}{\xi}^o_2 = \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1$, etc.

Subspace of symmetric tensors

Suppose that a physical problem involves a plane even if there are some non-planar aspects of the motion (e.g., oblique impact of a projectile onto a slab of armor). For solving the problem, any sensible engineer would line up a basis with the plane: all base vectors are either in the plane or normal to the plane.

The set of all symmetric tensors forms a *subspace*, which is analogous to a plane. The “normal” to the plane is the set of all skew-symmetric tensors. If you add two vectors in a plane, the result is also in the plane. Analogously, if you form any linear combination of symmetric tensors, the result is also symmetric.

Yield functions are defined for stress, which is symmetric. Our constitutive modelling problems intimately involve symmetric tensors, so it makes sense to use a basis for tensor space such that all base tensors are either purely symmetric or purely skew-symmetric.

Voigt vs. Mandel — Introduction

Voigt: $\{T\}^{\mathrm{v}} = \{T_{11}, T_{22}, T_{33}, T_{23}, T_{31}, T_{12}\}$

$$R_1^{\mathrm{v}} S_1^{\mathrm{v}} + R_2^{\mathrm{v}} S_2^{\mathrm{v}} + R_3^{\mathrm{v}} S_3^{\mathrm{v}} + \quad 2\left(R_4^{\mathrm{v}} S_4^{\mathrm{v}} + R_5^{\mathrm{v}} S_4^{\mathrm{v}} + R_6^{\mathrm{v}} S_6^{\mathrm{v}}\right)$$

Mandel: $\{T\}^{\mathrm{m}} = \{T_{11}, T_{22}, T_{33}, \sqrt{2}T_{23}, \sqrt{2}T_{31}, \sqrt{2}T_{12}\}$

Then $\underset{\approx}{R} : \underset{\approx}{S}$ means $R_1^{\mathrm{m}} S_1^{\mathrm{m}} + R_2^{\mathrm{m}} S_2^{\mathrm{m}} + R_3^{\mathrm{m}} S_3^{\mathrm{m}} + R_4^{\mathrm{m}} S_4^{\mathrm{m}} + R_5^{\mathrm{m}} S_4^{\mathrm{m}} + R_6^{\mathrm{m}} S_6^{\mathrm{m}}$

Q: Is the Mandel convention just a “trick” likely to bite us some day?

A: NO! Voigt components are the dangerous choice — they are referenced to an irregular basis for symmetric tensors. Mandel components are referenced to the same — *but normalized* — basis!

Change of basis for tensors

The basis expansion of *any* tensor may be written

$$\underset{\approx}{T} = T_{11}\underset{\sim}{e}_1\underset{\sim}{e}_1 + T_{12}\underset{\sim}{e}_1\underset{\sim}{e}_2 + T_{13}\underset{\sim}{e}_1\underset{\sim}{e}_3 + T_{21}\underset{\sim}{e}_2\underset{\sim}{e}_1 + T_{22}\underset{\sim}{e}_2\underset{\sim}{e}_2 + T_{23}\underset{\sim}{e}_2\underset{\sim}{e}_3 + T_{31}\underset{\sim}{e}_3\underset{\sim}{e}_1 + T_{32}\underset{\sim}{e}_3\underset{\sim}{e}_2 + T_{33}\underset{\sim}{e}_3\underset{\sim}{e}_3$$

$$= T_{(11)}\underset{\sim}{e}_1\underset{\sim}{e}_1 + T_{(22)}\underset{\sim}{e}_2\underset{\sim}{e}_2 + T_{(33)}\underset{\sim}{e}_3\underset{\sim}{e}_3$$
$$+ T_{(23)}(\underset{\sim}{e}_2\underset{\sim}{e}_3 + \underset{\sim}{e}_3\underset{\sim}{e}_2) + T_{(31)}(\underset{\sim}{e}_3\underset{\sim}{e}_1 + \underset{\sim}{e}_1\underset{\sim}{e}_3) + T_{(12)}(\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1)$$
$$+ T_{[32]}(\underset{\sim}{e}_3\underset{\sim}{e}_2 - \underset{\sim}{e}_2\underset{\sim}{e}_3) + T_{[13]}(\underset{\sim}{e}_1\underset{\sim}{e}_3 - \underset{\sim}{e}_3\underset{\sim}{e}_1) + T_{[21]}(\underset{\sim}{e}_2\underset{\sim}{e}_1 - \underset{\sim}{e}_1\underset{\sim}{e}_2)$$

where

$$T_{(ij)} \equiv \frac{1}{2}(T_{ij} + T_{ji}) \;\; and \;\; T_{[ij]} \equiv \frac{1}{2}(T_{ij} - T_{ji})$$

If the tensor is symmetric, the last three terms are all zero. If the tensor is skew-symmetric, then the first six terms are all zero and the last three terms are the components of the axial vector.

Voigt sym-dev basis

$$\underset{\approx}{T} = T_{(11)}\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1 + T_{(22)}\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2 + T_{(33)}\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3$$
$$+ T_{(23)}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2) + T_{(31)}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3) + T_{(12)}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2 + \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1)$$
$$+ T_{[32]}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2 - \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3) + T_{[13]}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3 - \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1) + T_{[21]}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1 - \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2)$$

Traditional Voigt:

$$T^{\mathrm{v}}_1 = T_{(11)},\ T^{\mathrm{v}}_2 = T_{(22)},\ T^{\mathrm{v}}_3 = T_{(33)},\ T^{\mathrm{v}}_4 = T_{(23)},\ T^{\mathrm{v}}_5 = T_{(31)},\ \ldots$$
$$\underset{\approx}{\xi}^{\mathrm{v}}_1 = \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1,\ \underset{\approx}{\xi}^{\mathrm{v}}_2 = \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2,\ \underset{\approx}{\xi}^{\mathrm{v}}_3 = \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3,\ \underset{\approx}{\xi}^{\mathrm{v}}_4 = (\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2),\ \underset{\approx}{\xi}^{\mathrm{v}}_5 = (\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3),\ \ldots$$

Then $\underset{\approx}{T} = \sum_{K=1}^{9} T^{\mathrm{v}}_K \underset{\approx}{\xi}^{\mathrm{v}}_K$.

For symmetric, $T_{(ij)} = T_{ij}$ and $T_{[ij]} = 0$.

MAJOR DISADVANTAGE: Voigt basis is not normalized!

Voigt basis is not normalized

Consider the inner product:

$$\underset{\approx}{R}:\underset{\approx}{S} = \left(\sum_{K=1}^{9} R^{\mathrm{v}}_K \underset{\approx}{\xi}^{\mathrm{v}}_K\right):\left(\sum_{J=1}^{9} S^{\mathrm{v}}_J \underset{\approx}{\xi}^{\mathrm{v}}_J\right) = \sum_{K=1}^{9}\sum_{J=1}^{9} R^{\mathrm{v}}_K S^{\mathrm{v}}_J (\underset{\approx}{\xi}^{\mathrm{v}}_K : \underset{\approx}{\xi}^{\mathrm{v}}_J)$$

The Voigt basis is orthogonal:

$\underset{\approx}{\xi}^{\mathrm{v}}_K : \underset{\approx}{\xi}^{\mathrm{v}}_J = 0$ if $K \neq J$.

The first three Voigt base tensors are normalized:

$\underset{\approx}{\xi}^{\mathrm{v}}_1 : \underset{\approx}{\xi}^{\mathrm{v}}_1 = 1$, $\underset{\approx}{\xi}^{\mathrm{v}}_2 : \underset{\approx}{\xi}^{\mathrm{v}}_2 = 1$, and $\underset{\approx}{\xi}^{\mathrm{v}}_3 : \underset{\approx}{\xi}^{\mathrm{v}}_3 = 1$, but the remaining base tensors are not normalized. They all have a magnitude of $\sqrt{2}$. Thus

$$\underset{\approx}{R}:\underset{\approx}{S} = \sum_{K=1}^{9} R^{\mathrm{v}}_K S^{\mathrm{v}}_K \left\|\underset{\approx}{\xi}_K\right\|^2 = R^{\mathrm{v}}_1 S^{\mathrm{v}}_1 + R^{\mathrm{v}}_2 S^{\mathrm{v}}_2 + R^{\mathrm{v}}_3 S^{\mathrm{v}}_3 + 2(R^{\mathrm{v}}_4 S^{\mathrm{v}}_4) + 2(R^{\mathrm{v}}_5 S^{\mathrm{v}}_5) + \ldots$$

MANDEL basis

Obvious thing to do ... normalize the basis.

Mandel basis: $\underset{\approx}{\xi}^{\mathrm{m}}_K \equiv \dfrac{\underset{\approx}{\xi}^{\mathrm{v}}_K}{\left\| \underset{\approx}{\xi}^{\mathrm{v}}_K \right\|}$.

$$T^{\mathrm{m}}_1 = T_{(11)},\ T^{\mathrm{m}}_2 = T_{(22)},\ T^{\mathrm{m}}_3 = T_{(33)},\ T^{\mathrm{m}}_4 = \sqrt{2}\,T_{(23)},\ T^{\mathrm{m}}_5 = \sqrt{2}\,T_{(31)},\ \ldots$$

$$\underset{\approx}{\xi}^{\mathrm{m}}_1 = \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1,\ \underset{\approx}{\xi}^{\mathrm{m}}_2 = \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2,\ \underset{\approx}{\xi}^{\mathrm{m}}_3 = \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3,\ \underset{\approx}{\xi}^{\mathrm{m}}_4 = \frac{(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2)}{\sqrt{2}},\ \underset{\approx}{\xi}^{\mathrm{m}}_5 = \frac{(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3)}{\sqrt{2}},\ \ldots$$

Then $\underset{\approx}{T} = \sum_{K=1}^{9} T^{\mathrm{m}}_K \underset{\approx}{\xi}^{\mathrm{m}}_K$, and $\underset{\approx}{\xi}^{\mathrm{m}}_K : \underset{\approx}{\xi}^{\mathrm{m}}_J = \delta_{KJ}$ and $\underset{\approx}{R} : \underset{\approx}{S} = \sum_{K=1}^{9} R^{\mathrm{v}}_K S^{\mathrm{v}}_K$

With this orthonormal Mandel basis, the tensor inner product takes a form that is a direct analog of the ordinary 3D vector inner product.

Mandel basis for symmetric tensors

The Mandel basis for 9D full tensor space is

$$\underset{\approx}{\xi}_1 = \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1, \qquad \underset{\approx}{\xi}_2 = \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2, \qquad \underset{\approx}{\xi}_3 = \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3$$

$$\underset{\approx}{\xi}_4 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2), \qquad \underset{\approx}{\xi}_5 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3), \qquad \underset{\approx}{\xi}_6 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2 + \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1)$$

$$\underset{\approx}{\xi}_7 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2 - \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3), \qquad \underset{\approx}{\xi}_8 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3 - \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1), \qquad \underset{\approx}{\xi}_9 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1 - \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2)$$

The basis is orthogonal because $\underset{\approx}{\xi}_K : \underset{\approx}{\xi}_J = 0$ if $K \neq J$. The basis is *normalized* (i.e., $\underset{\approx}{\xi}_K : \underset{\approx}{\xi}_J = \delta_{KJ}$) because of the factors of $\sqrt{2}$.

Just as an ordinary vector has components $v_k = \underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{e}}_k$, the Mandel components of a tensor $\underset{\approx}{T}$ are $T_K = \underset{\approx}{T} : \underset{\approx}{\xi}_K$.

Related topic: isomorphic stress space

Stress: $\underset{\approx}{\sigma}$

Mean stress: $p = \frac{1}{3}\mathrm{tr}\underset{\approx}{\sigma} = \frac{1}{3}\underset{\approx}{\mathbf{I}}:\underset{\approx}{\sigma}$ (positive in tension)

Stress deviator: $\underset{\approx}{\mathbf{S}} = \underset{\approx}{\sigma} - p\underset{\approx}{\mathbf{I}}$

Magnitude of the stress deviator: $\tau \equiv \sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}}$

Unit tensor in the direction of $\underset{\approx}{\mathbf{S}}$: $\hat{\underset{\approx}{\mathbf{S}}} \equiv \frac{\underset{\approx}{\mathbf{S}}}{\|\underset{\approx}{\mathbf{S}}\|} = \frac{\underset{\approx}{\mathbf{S}}}{\sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}}} = \frac{\underset{\approx}{\mathbf{S}}}{\tau}$

Then $\underset{\approx}{\sigma} = \tau\hat{\underset{\approx}{\mathbf{S}}} + p\underset{\approx}{\mathbf{I}}$.

We now show that non-intuitive factors appear because the identity $\underset{\approx}{\mathbf{I}}$ is not a unit tensor. Specificially, $\|\underset{\approx}{\mathbf{I}}\| = \sqrt{\underset{\approx}{\mathbf{I}}:\underset{\approx}{\mathbf{I}}} = \sqrt{3}$.

Motivational example

A popular simplified yield criterion assumes that the yield function depends only on τ and p. $F(\underset{\approx}{\sigma}) = f(\tau, p)$. The yield surface defined by $F(\underset{\approx}{\sigma}) = 0$ is a hypercylinder in stress space — it is a surface of revolution about the isotropic axis.

Gradient of yield: $\underset{\approx}{\mathbf{B}} = \dfrac{dF}{d\underset{\approx}{\sigma}} = \dfrac{\partial f}{\partial \tau}\left(\dfrac{d\tau}{d\underset{\approx}{\sigma}}\right) + \dfrac{\partial f}{\partial p}\left(\dfrac{dp}{d\underset{\approx}{\sigma}}\right) = \dfrac{\partial f}{\partial \tau}(\hat{\underset{\approx}{\mathbf{S}}}) + \dfrac{\partial f}{\partial p}\left(\dfrac{1}{3}\underset{\approx}{\mathbf{I}}\right)$

Let $\underset{\approx}{\sigma}^t = \tau^t \hat{\underset{\approx}{\mathbf{S}}} + p^t \underset{\approx}{\mathbf{I}}$ denote a trial elastic stress.

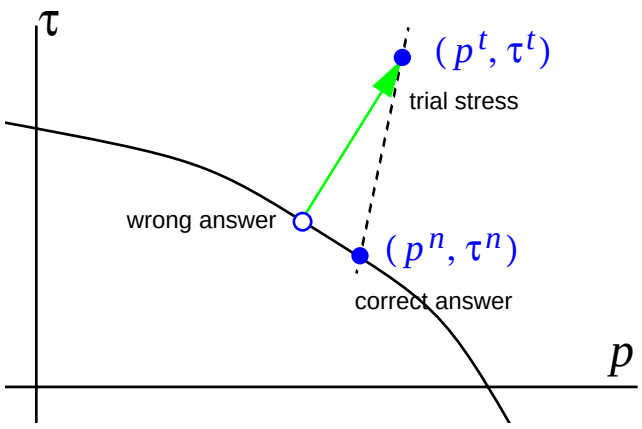
Let $\underset{\approx}{\sigma}^n = \tau^n \hat{\underset{\approx}{\mathbf{S}}} + p^n \underset{\approx}{\mathbf{I}}$ denote the new updated stress on the yield surface obtained by returning to the nearest point on the yield surace in stress space (which does not necessarily mean that the plastic strain rate is normal to the yield surface).

Normal projection (cont'd)

Return to nearest point on yield surface $\Rightarrow$ there's a scalar β such that

$$\underset{\approx}{\sigma}^t - \underset{\approx}{\sigma}^t = \beta \underset{\approx}{\mathbf{B}}, \quad \text{or} \quad (\tau^t - \tau^n)\hat{\underset{\approx}{\mathbf{S}}} + (p^t - p^n)\underset{\approx}{\mathbf{I}} = \beta\left(\frac{\partial f}{\partial \tau}(\hat{\underset{\approx}{\mathbf{S}}}) + \frac{\partial f}{\partial p}\left(\frac{1}{3}\underset{\approx}{\mathbf{I}}\right)\right)$$

Therefore $\dfrac{\tau^t - \tau^n}{p^t - p^n} = 3\left(\dfrac{\partial f/\partial \tau}{\partial f/\partial p}\right)$.



Thus, to project normal to the yield surface *in stress space*, you must project using a slope 3 times steeper than the normal in τ vs. p space. The problem is that the stress measures τ and p are not isomorphic to stress space. Viewed differently, the base tensors $\hat{\underset{\approx}{\mathbf{S}}}$ and $\underset{\approx}{\mathbf{I}}$ are orthogonal, but not normalized. We should use $\hat{\underset{\approx}{\mathbf{I}}} \equiv \dfrac{\underset{\approx}{\mathbf{I}}}{\|\underset{\approx}{\mathbf{I}}\|} = \dfrac{1}{\sqrt{3}}\underset{\approx}{\mathbf{I}}$ with an appropriately modified measure of mean stress. Namely, $\hat{p} = \sqrt{3}\,p$.

Rendulic plane

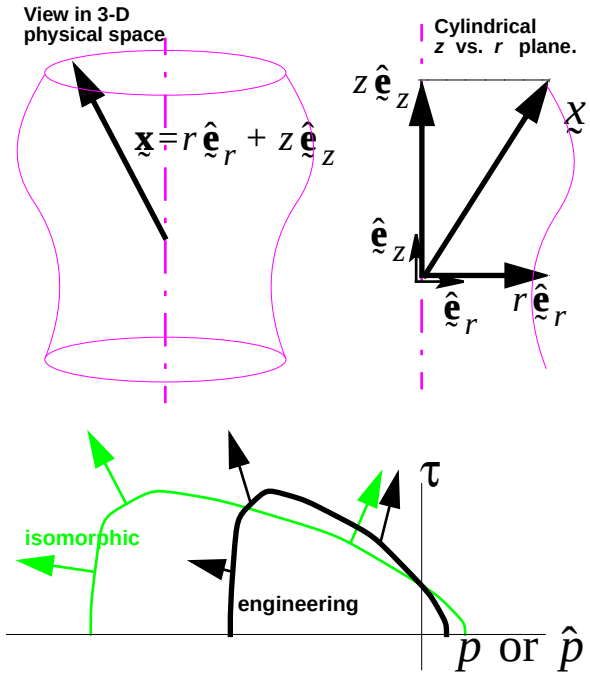
The Rendulic plane plots a “shear stress” versus a “mean stress.”

Engineer’s choice

“shear stress:” $\tau = \sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}}$, and

“mean stress:” $p = \frac{1}{3}\mathrm{tr}\underset{\approx}{\sigma}$. Then $\underset{\approx}{\sigma} = \underset{\approx}{\mathbf{S}} + p\underset{\approx}{I}$.

Problem: This τ vs. p space isn’t isomorphic to stress space. For example, $\underset{\approx}{\sigma}:\underset{\approx}{\sigma} \neq \tau^2 + p^2$. Importantly, the normal to the yield surface in τ vs. p space is *not* normal to the yield surface in stress space.



Mathematician’s choice: “shear stress” $\tau = \sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}} = \underset{\approx}{\sigma}:\hat{\underset{\approx}{\mathbf{S}}}$

“mean stress” $\hat{p} = \frac{1}{\sqrt{3}}\mathrm{tr}\underset{\approx}{\sigma} = \underset{\approx}{\sigma}:\hat{\underset{\approx}{\mathbf{I}}} = \sqrt{3}\,p$. Then $\underset{\approx}{\sigma} = \tau\hat{\underset{\approx}{\mathbf{S}}} + \hat{p}\hat{\underset{\approx}{I}}$. The normalized identity $\hat{\underset{\approx}{I}}$ is like the $\hat{\underset{\sim}{e}}_z$ cylindrical base vector.

Supplemental Topic: Anisotropic yield surfaces

For elastically anisotropic material, a very common "first-cut" best guess at the plastic yield surface is a Tsai-Wu ellipsoid of the form

$$f(\underset{\sim}{\sigma}) = (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*) : \underset{\approx}{\mathbf{L}} : (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*) - 1 ,$$ *(contrary to Walker's recent claims, this form is perfectly capable of modelling even highly anisotropic media.)*

where $\underset{\approx}{\mathbf{L}}$ shares the same anisotropy with the stiffness $\underset{\approx}{\mathbf{E}}$.

Elastic constants may be *nondestructively* measured, but the yield L_{ijkl} parameters are more difficult since a fresh sample must be used to measure each component. Thus, data are often lacking.

Proposal: Face with a dearth of data, assume that $\underset{\approx}{\mathbf{E}}$ and $\underset{\approx}{\mathbf{L}}$ have the same *eigenprojectors*, a term which we now define...

What are eigenprojectors?

To illustrate, consider simpler 3D space. Here's a sample tensor

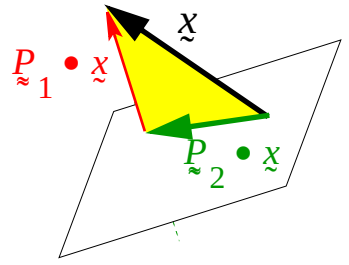
$$[\underset{\approx}{\mathbf{A}}] = \begin{bmatrix} 17 & -2 & -2 \\ -2 & 14 & -4 \\ -2 & -4 & 14 \end{bmatrix}, \text{ which has eigenpairs } \begin{aligned} \lambda_1 &= 9 & \underset{\sim}{\mathbf{v}}_1 &= \tfrac{1}{3}\{1, 2, 2\} \\ \lambda_2 &= 18 & \underset{\sim}{\mathbf{v}}_2 &= \tfrac{1}{\sqrt{5}}\{-2, 0, 1\} \\ \lambda_3 &= 18 & \underset{\sim}{\mathbf{v}}_3 &= \tfrac{1}{3\sqrt{5}}\{-2, 5, -4\} \end{aligned}$$

In spectral form, $\underset{\approx}{\mathbf{A}} = \lambda_1 \underset{\sim}{\mathbf{v}}_1 \underset{\sim}{\mathbf{v}}_1 + \lambda_2 \underset{\sim}{\mathbf{v}}_2 \underset{\sim}{\mathbf{v}}_2 + \lambda_3 \underset{\sim}{\mathbf{v}}_3 \underset{\sim}{\mathbf{v}}_3$

$$= 9\underbrace{\underset{\sim}{\mathbf{v}}_1 \underset{\sim}{\mathbf{v}}_1}_{\underset{\approx}{P}_1} + 18\underbrace{(\underset{\sim}{\mathbf{v}}_2 \underset{\sim}{\mathbf{v}}_2 + \underset{\sim}{\mathbf{v}}_3 \underset{\sim}{\mathbf{v}}_3)}_{\underset{\approx}{P}_2} \longleftarrow \text{unique!}$$

With respect to the principal basis,

$$\underset{\approx}{\mathbf{A}} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}, \quad \underset{\approx}{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad \underset{\approx}{P}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



http://www.me.unm.edu/~rmbrann/gobag.html

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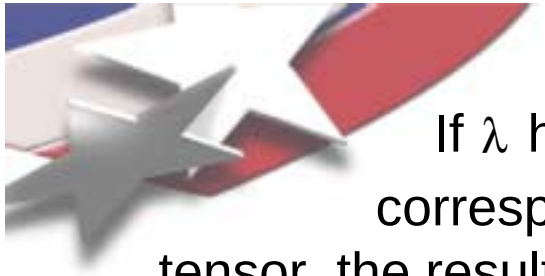
What are eigentensors?

We seek tensors $\underset{\approx}{Y}$ and scalars λ such that $\underset{\approx\approx}{\mathbf{E}} : \underset{\approx}{Y} = \lambda \underset{\approx}{Y}$. The major and minor symmetries of $\underset{\approx\approx}{\mathbf{E}}$ allow this to be written as an ordinary 6×6 matrix eigenproblem:

$$\begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & \sqrt{2}E_{1123} & \sqrt{2}E_{1131} & \sqrt{2}E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & \sqrt{2}E_{2223} & \sqrt{2}E_{2231} & \sqrt{2}E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & \sqrt{2}E_{3323} & \sqrt{2}E_{3331} & \sqrt{2}E_{3312} \\ \sqrt{2}E_{2311} & \sqrt{2}E_{2322} & \sqrt{2}E_{2333} & 2E_{2323} & 2E_{2331} & 2E_{2312} \\ \sqrt{2}E_{3111} & \sqrt{2}E_{3122} & \sqrt{2}E_{3133} & 2E_{3123} & 2E_{3131} & 2E_{3112} \\ \sqrt{2}E_{1211} & \sqrt{2}E_{1222} & \sqrt{2}E_{1233} & 2E_{1223} & 2E_{1231} & 2E_{1212} \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ \sqrt{2}Y_{23} \\ \sqrt{2}Y_{31} \\ \sqrt{2}Y_{23} \end{bmatrix} = \lambda \begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ \sqrt{2}Y_{23} \\ \sqrt{2}Y_{31} \\ \sqrt{2}Y_{23} \end{bmatrix}$$

An eigensolver will output a set of six orthonormal 6-dimensional eigenvectors. Each of these correspond to symmetric eigentensors.



If λ has multiplicity of 1, then $P_{ijkl} = Y_{ij}Y_{kl}$ is the corresponding eigenprojector. When it operates on any tensor, the result is the part of that tensor in the direction of Y_{ij}.

EXAMPLE: For isotropy, 3K is an eigenvalue of multiplicity 1. The *norrmalized* eigentensor is $\underset{\approx}{I}/\sqrt{3}$. The projector is $\frac{1}{3}\delta_{ij}\delta_{kl}$, which merely returns the isotropic part of any tensor it operates on.

If λ has multiplicity of 2, then the eigentensors $\underset{\approx}{Y}^{(1)}$ and $\underset{\approx}{Y}^{(2)}$ are not unique. Instead, the eigenprojector, $P_{ijkl} = Y_{ij}^{(1)}Y_{kl}^{(1)} + Y_{ij}^{(2)}Y_{kl}^{(2)}$ is unique. When it operates on an arbitrary tensor, the result is the part of the tensor in the subspace. Higher multiplicities are similar.

EXAMPLE: For isotropy, 2G is an eigenvalue of multiplicity 5. The eigenprojector (constructed by summing dyads of the five *orthonormalized* eigenprojectors) returns the deviator of any tensor it operates on. Thus, ANY DEVIATORIC TENSOR is an eigentensor for isotropy.

Back to anisotropic yield...

Recall $f(\underset{\sim}{\sigma}) = (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*) : \underset{\approx}{\mathbf{L}} : (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*) - 1$. If the material is transverse, the Mandel eigenproblem is of the form

$$\begin{bmatrix} E_0 & E_2 & E_3 & 0 & 0 & 0 \\ E_2 & E_0 & E_3 & 0 & 0 & 0 \\ E_3 & E_3 & E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_5 \end{bmatrix} \begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ \sqrt{2}Y_{23} \\ \sqrt{2}Y_{31} \\ \sqrt{2}Y_{23} \end{bmatrix} = \lambda \begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ \sqrt{2}Y_{23} \\ \sqrt{2}Y_{31} \\ \sqrt{2}Y_{23} \end{bmatrix},$$ where

$$E_1 = E_{3333},\ E_2 = E_{1122},\ E_3 = E_{1133},$$
$$E_4 = 2E_{2323},\ E_5 = 2E_{1212},\ E_o = E_2 + E_5$$

There are five independent stiffnesses, but only four independent eigenvalues (and therefore only four independent eigenprojectors). Forcing $\underset{\approx}{\mathbf{L}}$ to have the same eigenprojectors gives a formula for the elusive L_{1133} value that couples lateral and axial response.

Conclusions

This presentation covered many applications that illustrate the usefulness of regarding tensors as higher-dimensional vectors.

Key points were

- For radial and oblique return models, the stress may be returned to the yield surface via a projection operation that is analogous to projecting a simple vector onto a plane.
- Symmetric tensors are analogous to planes. The Mandel convention for symmetric tensor components correspond to an ortho*normal* basis for symmetric tensors.
- The invariant form of the Tresca yield criterion is invalid because negative values of that “yield function” do not necessarily correspond to stresses that are below yield.
- The isomorphic stress measures are a more accurate representation of stress space that is analogous to viewing the stress “vector” in the “plane” formed by the isotropic tensor and the stress itself.
- Anisotropic yield may be coupled to elastic isotropy via the elastic eigenprojectors.