Kinematics:
The mathematics of deformation

1. What is deformation?

Let Ω_o denote the initial configuration of a physical body (e.g., an airplane part, a blob of putty, etc.). Let $\underset{\sim}{X}$ denote the initial position of a point in the body. Disallowing fracture (where some particles "break" into two independent particles), each point in a body can have one and only one initial (pre-deformation) location, so the initial position vector, $\underset{\sim}{X}$, can be regarded as a unique "name" of any particular particle of interest.

Let Ω denote the configuration of the body after deformation. In general, each material particle $\underset{\sim}{X}$ moves to a new location, $\underset{\sim}{x}$, and no two distinct particles are allowed to deform into the same location.

The drawing below shows a *pseudo* 2D deformation in which a square grid has been conceptually "painted" onto the material. We have filled in a few of the material elements (grid cells) with solid color to help us later. Of course, since the out-of-plane thickness has changed (by about 50% from the looks of it), this deformation is actually three-dimensional. A pseudo 2D deformation has out-of-plane thickness changes, but no out-of-plane shearing; motion is characterized principally with respect to the two in-plane coordinates.

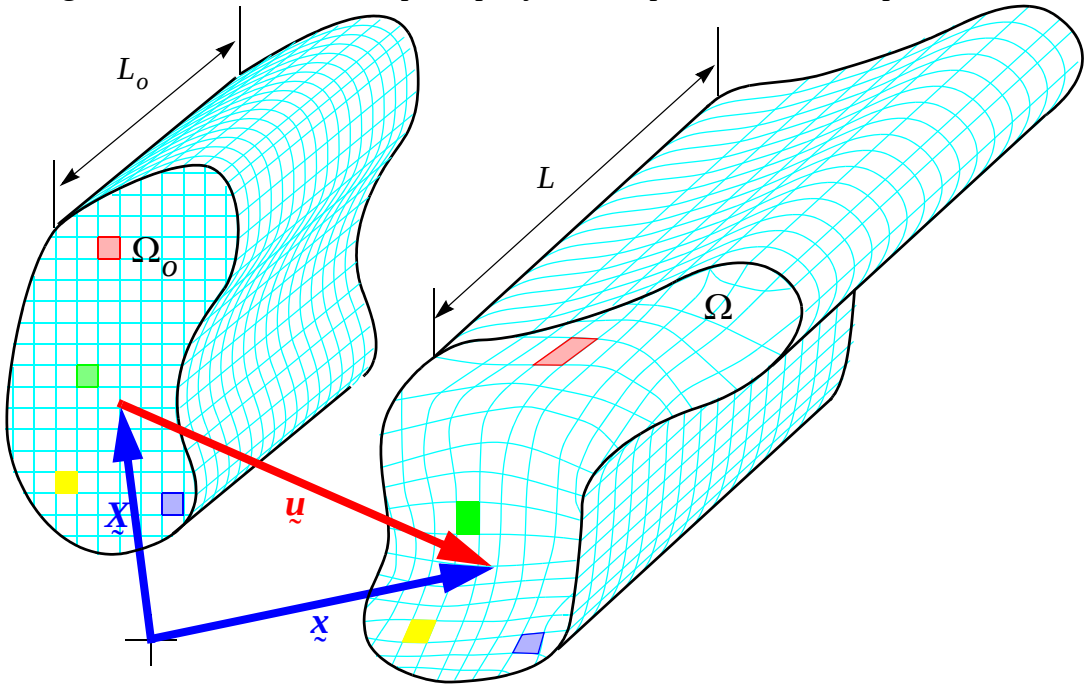


Figure 1.1. A general 2D deformation. A slice of putty is deformed into a new configuration in pseudo two-dimensional manner. The cross-section and the out-of plane length change arbitrarily, but no out-of-plane *shearing* is allowed. In this sketch, a grid has been conceptually "painted" onto the material to facilitate visualizing the deformation.

Most of the concepts in this document will be illustrated in the context of pseudo 2D deformations, but they apply equally well to general deformations in which material elements can shear and rotate in arbitrary directions.

Deformation

Since each particle occupies its own unique location in space in both the initial and deformed configurations, we may assert that there exists a one-to-one function χ (called a mapping function) such that

$$\underset{\sim}{x} = \chi(\underset{\sim}{X}) \tag{1.1}$$

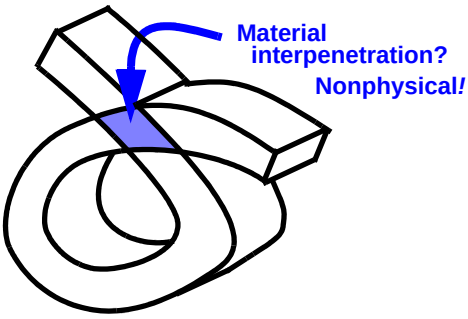


Figure 1.2. A locally admissible, but globally inadmissible deformation.

Examples of mapping functions are given in Section 5. Naturally, no single material element (i.e., no grid cell in Fig. 1.1) should be permitted to "invert." This constraint is called "**local admissibility**" of the deformation, and it is ensured if the matrix $[\partial x_i/\partial X_j]$ has a positive determinant. In finite element simulations, a technique called "hourglass control" is typically used to ensure local admissibility. However, local admissibility is not sufficient to guarantee "**global admissibility**," which requires that the mapping function in Eq. (1.1) be invertible and therefore prohibits material interpenetration. Figure 1.2, for example, shows a globally inadmissible deformation that is everywhere locally admissible. In computational mechanics, "contact algorithms" must be used to avoid material the globally inadmissible phenomenon of material interpenetration. Later on, after we define the deformation gradient tensor, $\underset{\approx}{\boldsymbol{F}}$, whose ij components are given by $[\partial x_i/\partial X_j]$. Occasionally, problems in mechanics involve specification only of how the $\underset{\approx}{\boldsymbol{F}}$ tensor varies in space without making explicit reference to the mapping function, $\underset{\sim}{x} = \chi(\underset{\sim}{X})$. In these problems, another admissibility condition (called the **compatibility condition**) requires that at least the mapping function *must exist* — the $\underset{\approx}{\boldsymbol{F}}$ tensor field must be integrable even if we choose not to integrate it.

Displacement

The **displacement** of a particle is a vector extending from a particle's initial location to its deformed location (see Fig. 1.1):

$$\underset{\sim}{\boldsymbol{u}} \equiv \underset{\sim}{x} - \underset{\sim}{X} \tag{1.2}$$

The initial and final position vectors, $\underset{\sim}{\boldsymbol{x}}$ and $\underset{\sim}{\boldsymbol{X}}$, are origin-dependent (they depend on where you place the origin). The displacement vector $\underset{\sim}{\boldsymbol{u}}$ is a *free* vector — it won't change if you move the origin. The definitions of positions and displacement can be generalized to permit different origins to be used in the initial and deformed configurations. If the deformed origin is located by extending a vector $\underset{\sim}{\boldsymbol{c}}$ from the initial origin, then $\underset{\sim}{\boldsymbol{x}}$ in the above expressions must be replaced by $\underset{\sim}{\boldsymbol{c}} + \underset{\sim}{\boldsymbol{x}}$. The origin-to-origin vector $\underset{\sim}{\boldsymbol{c}}$ does not vary with position — it is spatially constant.

2. The deformation gradient

The deformation gradient tensor (graphical introduction)

GOAL: Provide a physical understanding of the deformation gradient tensor prior to defining it mathematically.

The sketch of deformation (Fig. 1.1) shows a square grid conceptually "painted" onto the undeformed body, and some of the grid cells have been filled in with solid color to help us better visualize the deformation. Note that each little material square deforms to a shape that is close to a parallelogram in shape. This approximation (squares-to-parallelograms) becomes exact in the limit of an infinitely refined grid. In three dimensions, *cubes* deform into *parallelepipeds*. On surfaces embedded in 3D space, infinitesimal squares deform into infinitesimal parallelograms. The deformation gradient is a tensor that quantifies both the 3D and 2D shape change as well as overall material rotation, making it superior to strain as an all-encompassing measure of deformation of material elements. If you have the deformation gradient tensor, then you can compute the strain; the converse is not true. The purpose of this section is to describe how to graphically approximate the deformation gradient tensor by simply *looking* at the deformation. We will postpone giving an exact mathematical definition of the deformation gradient tensor.

Any cube can be characterized by the three orthonormal vectors that form its edges. Likewise, the *deformed* parallelepiped can be characterized by the three vectors that form its edges. The deformation gradient tensor $\underset{\approx}{\boldsymbol{F}}$ quantifies the changes in these edge vectors by assembling the three deformed edge vectors into columns of a 3×3 matrix of components. For rectangular Cartesian coordinate systems, the columns of the deformation gradient's component matrix contain the deformed parallelepiped edge vectors, expressed relative to the undeformed edge vectors that defined the initial cube. By *relative,* we mean that all length changes are expressed as multiples of the initial cube's edge lengths, and all directions are expressed relative to the initial cube's edge directions. Consequently (by appropriately defining our unit of length) we may pretend that the initial infinitesimal cube is a *unit* cube whose three edge vectors are aligned with the orthonormal "laboratory" basis, $\{\underset{\sim}{\boldsymbol{E}}_1, \underset{\sim}{\boldsymbol{E}}_2, \underset{\sim}{\boldsymbol{E}}_3\}$. Upon deformation, these edge vectors deform into a new set of vectors, $\{\underset{\sim}{\boldsymbol{g}}_1, \underset{\sim}{\boldsymbol{g}}_2, \underset{\sim}{\boldsymbol{g}}_3\}$, that define the edges of the deformed parallelepiped. If you find the lab components of the i^{th} deformed edge vector and assemble those components into the i^{th} column of $\underset{\approx}{\boldsymbol{F}}$, then the result will be the component matrix for $\underset{\approx}{\boldsymbol{F}}$ with respect to the $\{\underset{\sim}{\boldsymbol{E}}_1, \underset{\sim}{\boldsymbol{E}}_2, \underset{\sim}{\boldsymbol{E}}_3\}$ (lab) basis. Mathematically, this means that

$$\underset{\sim}{g}_k = \underset{\approx}{F} \bullet \underset{\sim}{E}_k \tag{2.1}$$

or, equivalently,

$$\underset{\sim}{g}_k = F_{jk}\underset{\sim}{E}_j, \tag{2.2}$$

where F_{jk} are the jk components of $\underset{\approx}{F}$ with respect to the lab basis.* The k^{th} column of $[F_{jk}]$ contains the lab components of $\underset{\sim}{g}_k$. These equations provide us with a method for approximating the local deformation gradient at a point whenever we have accurate pictures of the deformed cube (or, for 2D, the deformed square).

The deformed $\underset{\sim}{g}_k$ edge vectors are not generally orthogonal or of unit length. They are called "material vectors" because they "travel along" with the same set of material points (rotating with and stretching in proportion to the grid lines). In general, if $\underset{\sim}{M}$ is a material vector in the initial configuration (not necessarily of unit length), then it will flow with the material to become a new (stretched and rotated) vector $\underset{\sim}{m}$ in the deformed configuration, given by

$$\underset{\sim}{m} = \underset{\approx}{F} \bullet \underset{\sim}{M}, \text{ and the "fiber stretch" is defined } \lambda = ||\underset{\sim}{m}||/||\underset{\sim}{M}||. \tag{2.3}$$

Thus, if the $\underset{\approx}{F}$ tensor is constructed from knowledge of how the unit lab base vectors would flow with material at a point, then you can immediately compute how *any* material vector (not necessarily aligned with the lab basis and not necessarily of unit length) would flow with the material. This is the fundamental characteristic of a *linear* transformation. We will explain later that $F_{ij} = \partial x_i/\partial X_j$. The mapping function $\underset{\sim}{x} = \chi(\underset{\sim}{X})$ is generally nonlinear, but its derivative at a point is linear with respect to small changes in position. This statement is analogous to what a freshman calculus student encounters when first introduced to the concept of differentiation. A function $y = f(x)$ might be nonlinear, but one can always (for smooth functions) define a straight (i.e., linear) line that is tangent to the curve at any point and the slope of the line is given by dy/dx. The deformation gradient tensor, $F_{ij} = \partial x_i/\partial X_j$, plays a role in three dimensions that is analogous to the tangent slope dy/dx in one dimension. The difficulty lies in visualization. In one dimension, we can always plot the function $y = f(x)$ and then show the tangent line on the graph. In three dimensions, we must understand the mapping function $\underset{\sim}{x} = \chi(\underset{\sim}{X})$ in a more abstract manner by talking about what it does (locally) to little cubes of material. Our perspective

* Comparing Eqs. (2.1) and (2.2), you might (wrongly) think that we should have written F_{kj} instead of F_{jk} in Eq. (2.2). To see why Eq. Eq. (2.2) is correct, note that any vector $\underset{\sim}{v}$ may be expressed in terms of the lab basis as $\underset{\sim}{v} = v_k\underset{\sim}{E}_k$, where $v_k = \underset{\sim}{v} \bullet \underset{\sim}{E}_k$. Stated differently, any vector $\underset{\sim}{v}$ may be written $\underset{\sim}{v} = (\underset{\sim}{v} \bullet \underset{\sim}{E}_k)\underset{\sim}{E}_k$, or, since the dot product is commutative, $\underset{\sim}{v} = (\underset{\sim}{E}_k \bullet \underset{\sim}{v})\underset{\sim}{E}_k$. This identity applies for any vector $\underset{\sim}{v}$. Applying it to one of the stretched edge vectors, $\underset{\sim}{g}_k$, gives $\underset{\sim}{g}_k = (\underset{\sim}{E}_i \bullet \underset{\sim}{g}_k)\underset{\sim}{E}_i$, or, substituting Eq. (2.1), $\underset{\sim}{g}_k = F_{ik}\underset{\sim}{E}_i$, where $F_{ik} = \underset{\sim}{E}_i \bullet \underset{\approx}{F} \bullet \underset{\sim}{E}_k$.

here would be analogous to describing the 1D mapping $y = f(x)$ by stating that a little line segment dx deforms into a new line segment dy that differs in length by a ratio $s = dy/dx$ so that $dy = s dx$. For generalization into 3D mapping, this last equation becomes $d\underset{\sim}{x} = \underset{\approx}{F} \bullet d\underset{\sim}{X}$.

Before refining this loose mathematical discussion, let's spend some time constructing some deformation gradient tensors graphically. What we will be doing here is analogous to taking out a ruler and estimating the local slope of a 1D curve $y = f(x)$ by measuring local "rise over run."

EXAMPLES

Consider the top material element sketched in Fig. 2.1. Zooming in on the undeformed and deformed shapes of these elements, and introducing a "ruler" for which the initial cube has unit length gives the following drawing:

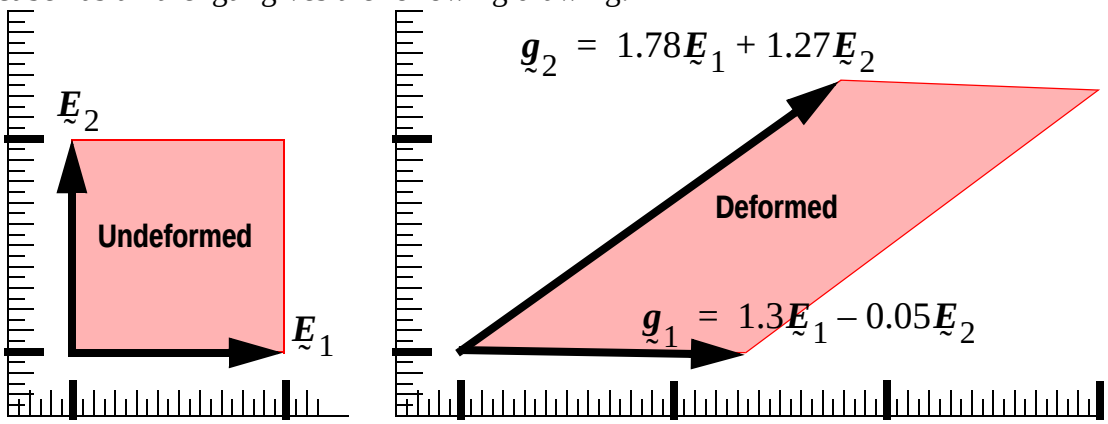


Figure 2.1. A graphical determination of the deformation gradient This sketch shows a zoomed in picture of the small material element at the top of Figure 1.1 on page 1. A "ruler" has been drawn on this figure so that the undeformed square is a unit square.

Using the rulers as a guide, we can "read off" the components of each of the deformed edge vectors to obtain:

$$\{\underset{\sim}{g}_1\} = \begin{Bmatrix} 1.3 \\ -0.05 \end{Bmatrix} \qquad \text{and} \qquad \{\underset{\sim}{g}_2\} = \begin{Bmatrix} 1.78 \\ 1.27 \end{Bmatrix} \tag{2.4}$$

Assembling these components into *columns* of a 2×2 matrix gives

$$[\underset{\approx}{F}] = \begin{bmatrix} 1.3 & 1.78 \\ -0.05 & 1.27 \end{bmatrix} \tag{2.5}$$

The fiber stretches are

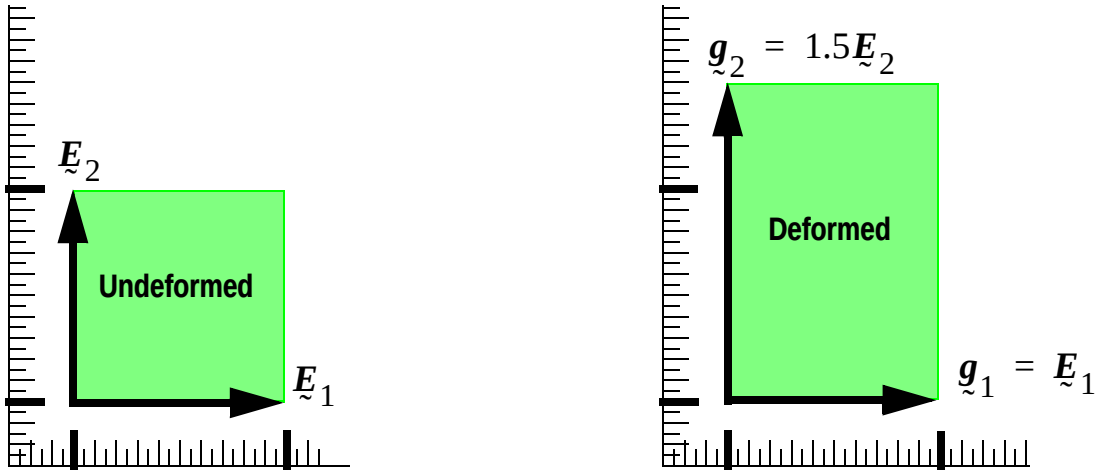
$$\lambda_1 = \|\underset{\sim}{g}_1\| / \|\underset{\sim}{E}_1\| = 1.3 \qquad \text{and} \qquad \lambda_2 = \|\underset{\sim}{g}_2\| / \|\underset{\sim}{E}_2\| = 2.2 \tag{2.6}$$

which mean that an infinitesimal fiber originally aligned with $\underset{\sim}{E}_1$ will, after deformation, have a length 1.3 times as large. A fiber originally aligned with $\underset{\sim}{E}_2$ will change length by a factor of 2.2.

The deformation in Fig. 1.1 is actually three dimensional — the out-of-plane thickness has increased by about 50%, which corresponds to an out-of-plane stretch (L/L_o) equal to 1.5. Consequently, the $\underset{\sim}{\boldsymbol{E}}_3$ base vector has deformed to become $\underset{\sim}{\boldsymbol{g}}_3 = 1.5\underset{\sim}{\boldsymbol{E}}_3$. Consequently, the pseudo 2D deformation gradient in Eq. (2.5) can be "upgraded" to 3D as

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} 1.3 & 1.78 & 0 \\ -0.05 & 1.27 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \tag{2.7}$$

If we zoom in on the middle element in our original sketch and graphically measure the deformed material element edge vectors, then the deformation gradient is found as follows:

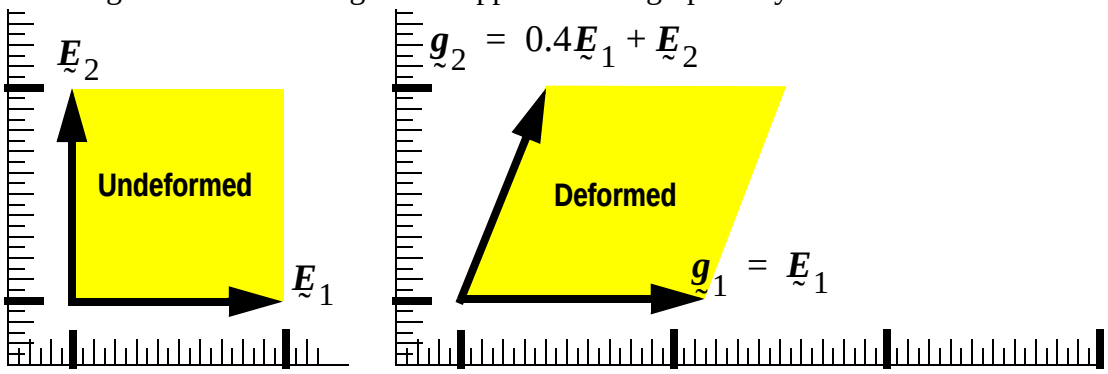


Component arrays:

$$\{\underset{\sim}{\boldsymbol{g}}_1\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \{\underset{\sim}{\boldsymbol{g}}_2\} = \begin{Bmatrix} 0 \\ 1.5 \end{Bmatrix} \qquad [\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}$$

Figure 2.2. A graphical approximation of the local deformation gradient This figure shows an enlarged picture of the middle material element shown in Figure 1.1 on page 1. This material element happens to be in a state of uniaxial strain in the 2-direction.

Finally, the material element in the lower left of Figure 1.1 on page 1 is deformed according to a deformation gradient approximated graphically as follows:



Component arrays:

$$\{\underset{\sim}{g}_1\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \{\underset{\sim}{g}_2\} = \begin{Bmatrix} 0.4 \\ 1 \end{Bmatrix} \qquad [\underset{\approx}{F}] = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix}$$

Figure 2.3. Graphical determination of the deformation gradient for the bottom element in Fig. 1.1

This material element coincidentally appears to be in a state of pure shear.

Physically interpreting a deformation gradient tensor

Consider the inverse problem: suppose you know the deformation gradient (e.g., as output from a structural mechanics finite element code), and you wish to visualize the nature of the deformation by sketching the material element. The process is simple. Just interpret the i^{th} column of $[\underset{\approx}{\boldsymbol{F}}]$ to contain the lab components of the deformed edge vectors, and then draw the deformed element accordingly. Suppose, for example, that you are given

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} \frac{1}{2} & 2 \\ -1 & -\frac{1}{4} \end{bmatrix} \tag{2.8}$$

The first column contains $\underset{\sim}{\boldsymbol{g}}_1$ and the second column contains $\underset{\sim}{\boldsymbol{g}}_2$. Specifically,

$$\{\underset{\sim}{\boldsymbol{g}}_1\} = \begin{Bmatrix} 1/2 \\ -1 \end{Bmatrix} \quad \text{and} \quad \{\underset{\sim}{\boldsymbol{g}}_2\} = \begin{Bmatrix} 2 \\ 1/4 \end{Bmatrix} \tag{2.9}$$

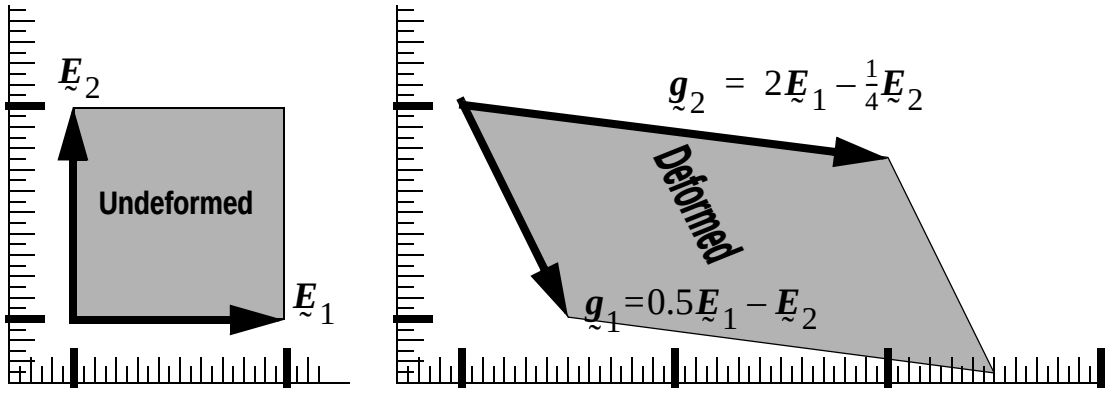


Figure 2.4. Sketching the deformation when the deformation gradient is known This deformation corresponds to the deformation gradient in Eq. (2.8).

Being only a collection of four numbers, the deformation gradient matrix in Eq. (2.9) does not have much obvious physical meaning by itself. But, by drawing the associated deformation, we can now see that this deformation is essentially a movement of the material consisting of stretching it in the 2-direction, then shearing it in the 1-direction, and finally rotating it clockwise.

Sequentially applied deformations

Suppose that some total deformation $\underset{\approx}{\boldsymbol{F}}$ is achieved via two separate deformation phases, the first being defined by one deformation gradient tensor $\underset{\approx}{\boldsymbol{F}}^{(1)}$ and the next defined by $\underset{\approx}{\boldsymbol{F}}^{(2)}$. At the end of the first phase, a material vector $\underset{\sim}{\boldsymbol{M}}$ will stretch and rotate into a new vector

$$\underset{\sim}{\boldsymbol{m}}^{\text{intermediate}} = \underset{\approx}{\boldsymbol{F}}^{(1)} \bullet \underset{\sim}{\boldsymbol{M}} \tag{2.10}$$

At the end of the second phase, this vector will become

$$\underset{\sim}{\boldsymbol{m}} = \underset{\approx}{\boldsymbol{F}}^{(2)} \bullet \underset{\sim}{\boldsymbol{m}}^{\text{intermediate}} = \underset{\approx}{\boldsymbol{F}}^{(2)} \bullet \underset{\approx}{\boldsymbol{F}}^{(1)} \bullet \underset{\sim}{\boldsymbol{M}} \tag{2.11}$$

Thus, the overall final deformation gradient tensor is

$$\boxed{\underset{\approx}{\boldsymbol{F}} = \underset{\approx}{\boldsymbol{F}}^{(2)} \bullet \underset{\approx}{\boldsymbol{F}}^{(1)}} \tag{2.12}$$

Note that the tensors are multiplied in reverse order of application. Changing the order of application will result in a different material deformation (See Fig. 2.5).

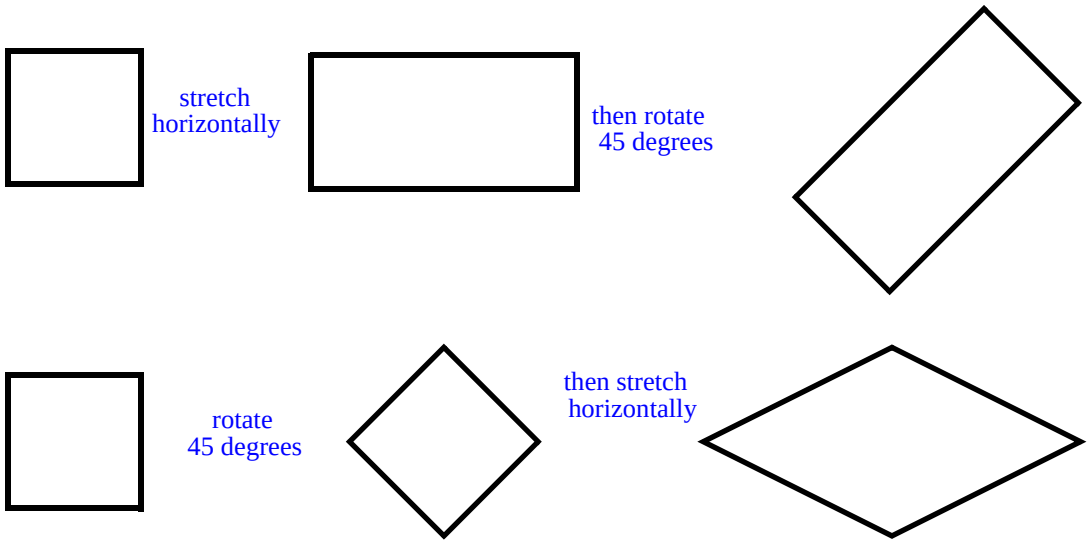


Figure 2.5. Illustration that order of application matters!

Tracking volume changes

A volume element in the undeformed configuration can be defined by the parallelepiped formed by of any three (infinitesimal) material edge vectors, $\{\underset{\sim}{\boldsymbol{M}}_1, \underset{\sim}{\boldsymbol{M}}_2, \underset{\sim}{\boldsymbol{M}}_3\}$. The undeformed volume of this infinitesimal material element is given by the triple scalar product:

$$V_o = [\underset{\sim}{\boldsymbol{M}}_1, \underset{\sim}{\boldsymbol{M}}_2, \underset{\sim}{\boldsymbol{M}}_3] \tag{2.13}$$

The initial material vectors each stretch and rotate, becoming new deformed vectors, $\{\underset{\sim}{\boldsymbol{m}}_1, \underset{\sim}{\boldsymbol{m}}_2, \underset{\sim}{\boldsymbol{m}}_3\}$, that define the deformed shape of the material element. The deformed volume is given by

$$V = [\underset{\sim}{\boldsymbol{m}}_1, \underset{\sim}{\boldsymbol{m}}_2, \underset{\sim}{\boldsymbol{m}}_3] \tag{2.14}$$

or, noting that $\underset{\sim}{\boldsymbol{m}}_k = \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{M}}_k$,

$$V = [\underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{M}}_1, \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{M}}_2, \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{M}}_3] \tag{2.15}$$

Expanding this out in indicial notation and using the definition of a determinant reveals that

$$V = (\det \underset{\approx}{\boldsymbol{F}})[\underset{\sim}{\boldsymbol{M}}_1, \underset{\sim}{\boldsymbol{M}}_2, \underset{\sim}{\boldsymbol{M}}_3] = \det \underset{\approx}{\boldsymbol{F}} V_o, \tag{2.16}$$

or

$$\boxed{\frac{V}{V_o} = \det \underset{\approx}{\boldsymbol{F}}} \tag{2.17}$$

where J is called the "**Jacobian**" of the deformation, defined by

$$\boxed{J \equiv \det \underset{\approx}{\boldsymbol{F}}} \tag{2.18}$$

In words, the determinant of the deformation gradient tensor equals the ratio of the deformed to the undeformed infinitesimal volume elements. For realistic deformations, both volume elements must be positive. Therefore, we may assert that

$$J > 0 \tag{2.19}$$

Therefore, the deformation gradient tensor must be invertible. This is the mathematical statement of local admissibility, discussed on page 2.

Tracking *surfaces*

The $\underset{\sim}{\boldsymbol{g}}_k$ vectors move with the material. They always lie along the same set of points (more correctly, they're always tangent to and stretch with the grid lines). In three dimensions, the triad $\{\underset{\sim}{\boldsymbol{g}}_1, \underset{\sim}{\boldsymbol{g}}_2, \underset{\sim}{\boldsymbol{g}}_3\}$ forms the edges of a parallelepiped that defines the deformed shape of the material element. Upon deformation, a *face* of the parallelepiped is defined by the two material edge vectors that form that face. If, for example, we consider the face formed by $\underset{\sim}{\boldsymbol{g}}_2$ and $\underset{\sim}{\boldsymbol{g}}_3$, then the normal to this face is proportional to $\underset{\sim}{\boldsymbol{g}}_2 \times \underset{\sim}{\boldsymbol{g}}_3$

Even though the edge vectors move with the material, the outward normals to planes do *not* move with the material. Consider the deformation shown in Fig. 2.4, for which the deformation gradient is

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} \frac{1}{2} & 2 \\ -1 & -\frac{1}{4} \end{bmatrix} \tag{2.20}$$

To extend this example to three dimensions (so that we can talk about *planes*), let's suppose that the deformation is actually that of a thick plate (no stretching occurs in the out-of-plane direction), so the 3D deformation gradient is

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} \frac{1}{2} & 2 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.21}$$

We have already explained that $\underset{\sim}{\boldsymbol{g}}_k$ comes from the k^{th} column of $\underset{\approx}{\boldsymbol{F}}$, which means

$$\underset{\sim}{\boldsymbol{g}}_k = \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{E}}_k \tag{2.22}$$

Now we introduce complementary vectors $\underset{\sim}{\nu}_k$ defined by

$$\underset{\sim}{\nu}_i = \underset{\approx}{\boldsymbol{F}}^{-T} \bullet \underset{\sim}{\boldsymbol{E}}_i \tag{2.23}$$

Note that

$$\underset{\sim}{\nu}_i \bullet \underset{\sim}{\boldsymbol{g}}_k = \delta_{ik} \tag{2.24}$$

Incidentally, because the vector $\underset{\sim}{\nu}_i$ obeys Eq. (2.24), it is called the "dual" or "contravariant" vector associated with $\underset{\sim}{\boldsymbol{g}}_k$ and, in publications that employ convected coordinates, the vector $\underset{\sim}{\nu}_i$ is typically denoted $\underset{\sim}{\boldsymbol{g}}^i$ (which is distinguished from $\underset{\sim}{\boldsymbol{g}}_i$ because it uses a superscript rather than a subscript). We will continue to use the symbol $\underset{\sim}{\nu}_i$ for the dual vector.

Suppose, for example, $\underset{\sim}{\nu}_1$ is perpendicular to both $\underset{\sim}{\boldsymbol{g}}_2$ and $\underset{\sim}{\boldsymbol{g}}_3$; hence, it must be perpendicular to the material surface formed by $\underset{\sim}{\boldsymbol{g}}_2$ and $\underset{\sim}{\boldsymbol{g}}_3$. As far as the normal goes, the magnitude of $\underset{\sim}{\nu}_1$ is inconsequential and may be chosen for computational convenience. Specifically, our definition in Eq. (2.23) corresponds to

$$\underset{\sim}{\nu}_1 = \frac{(\underset{\sim}{\boldsymbol{g}}_2 \times \underset{\sim}{\boldsymbol{g}}_3)}{g_o}, \text{ where } g_o \equiv \underset{\sim}{\boldsymbol{g}}_1 \bullet (\underset{\sim}{\boldsymbol{g}}_2 \times \underset{\sim}{\boldsymbol{g}}_3) \tag{2.25}$$

In three dimensions

$$[\underset{\approx}{\boldsymbol{F}}^{-T}] = \frac{2}{15}\begin{bmatrix} -1 & 4 & 0 \\ -8 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.26}$$

The columns of this matrix contain the lab components of the $\underset{\sim}{\boldsymbol{\nu}}_k$ vectors, sketched in Fig. 2.6. This picture shows quite clearly that normals to material planes are not material vectors — the don't flow with the material.

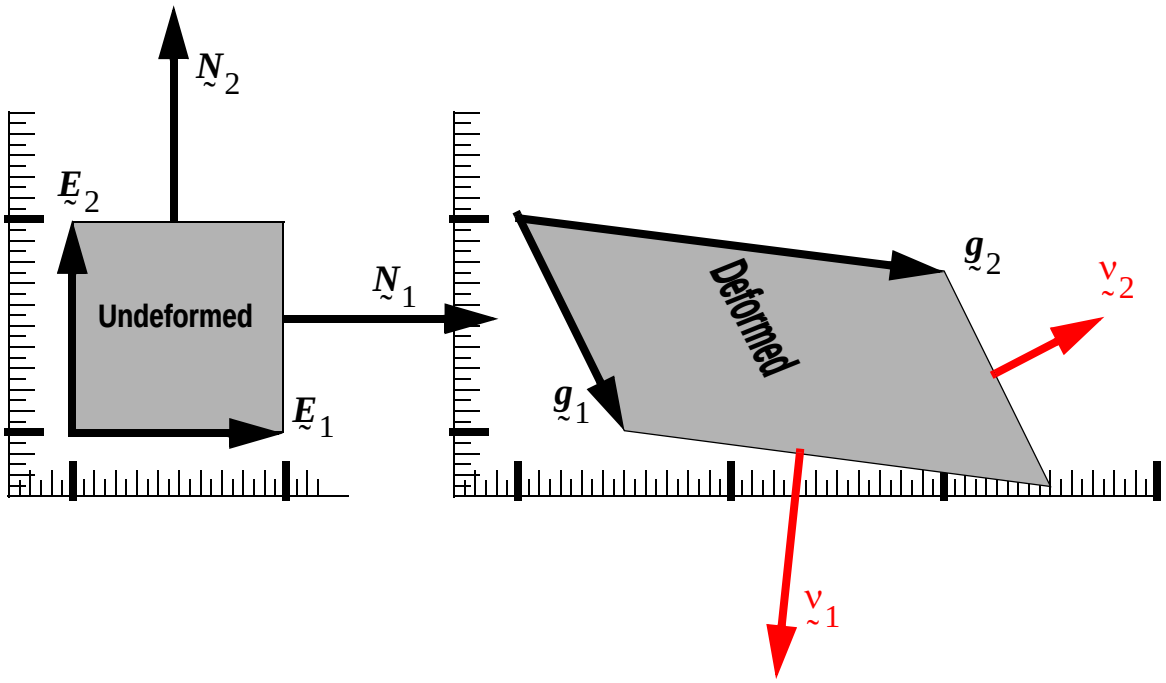


Figure 2.6. Tracking material surfaces This deformation corresponds to the deformation gradient in Eq. (2.8). The deformed edges are coincident with $\underset{\sim}{\boldsymbol{g}}_i = \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{E}}_i$ outward normals to the material surfaces are parallel to $\underset{\sim}{\nu}_i = \underset{\approx}{\boldsymbol{F}}^{-T} \bullet \underset{\sim}{\boldsymbol{E}}_i$.

In general, if a material plane (i.e., one that moves with the material) has a normal $\underset{\sim}{\boldsymbol{N}}$ (not required to be of unit length) in the undeformed configuration, then after deformation, the vector

$$\underset{\sim}{\nu} = \underset{\approx}{\boldsymbol{F}}^{-T} \bullet \underset{\sim}{\boldsymbol{N}} \tag{2.27}$$

(again not generally of unit length) will be normal to the plane in the deformed configuration. If the unit normal is desired, then this result can simply be normalized.

Area vectors

Let $\underset{\sim}{\boldsymbol{M}}_1$ and $\underset{\sim}{\boldsymbol{M}}_2$ be any two material vectors in the undeformed configuration. The area of the parallelogram formed by these two vectors is given by

$$\underset{\sim}{\boldsymbol{A}}_o = \underset{\sim}{\boldsymbol{M}}_1 \times \underset{\sim}{\boldsymbol{M}}_2 \tag{2.28}$$

If the material vectors are infinitesimal, then this area vector will be infinitesimal. Upon deformation, the $\underset{\sim}{\boldsymbol{M}}_k$ vectors deform into new stretched and rotated vectors, $\underset{\sim}{\boldsymbol{m}}_1$ and $\underset{\sim}{\boldsymbol{m}}_2$, that define a new area vector

$$\underset{\sim}{\boldsymbol{A}} = \underset{\sim}{\boldsymbol{m}}_1 \times \underset{\sim}{\boldsymbol{m}}_2 \tag{2.29}$$

Substituting Eq. (2.3) into this expression gives

$$\underset{\sim}{\boldsymbol{A}} = (\underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{M}}_1) \times (\underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{M}}_2) \tag{2.30}$$

Using an identity from tensor analysis, this result can be written as

$$\underset{\sim}{\boldsymbol{A}} = \underset{\approx}{\boldsymbol{F}}^c \bullet (\underset{\sim}{\boldsymbol{M}}_1 \times \underset{\sim}{\boldsymbol{M}}_2) \tag{2.31}$$

or

$$\underset{\sim}{\boldsymbol{A}} = \underset{\approx}{\boldsymbol{F}}^c \bullet \underset{\sim}{\boldsymbol{A}}_o \tag{2.32}$$

where $\underset{\approx}{\boldsymbol{F}}^c$ is the *cofactor* of $\underset{\approx}{\boldsymbol{F}}$. Since the deformation gradient is invertible, the cofactor may be alternatively written

$$\underset{\approx}{\boldsymbol{F}}^c = (\det \underset{\approx}{\boldsymbol{F}}) \underset{\approx}{\boldsymbol{F}}^{-T} \tag{2.33}$$

Recalling that $J = \det \underset{\approx}{\boldsymbol{F}}$ and also recalling that $\underset{\sim}{\boldsymbol{M}}_1 \times \underset{\sim}{\boldsymbol{M}}_2$ is just the undeformed area vector, $\underset{\sim}{\boldsymbol{A}}_o$, Eq. (2.32) may be written

$$\underset{\sim}{\boldsymbol{A}} = J \underset{\approx}{\boldsymbol{F}}^{-T} \bullet \underset{\sim}{\boldsymbol{A}}_o \tag{2.34}$$

This famous result is called “Nanson’s relation.”

Deformation gradient (precise mathematical definition)

So far, we have described a *graphical* method of approximating the deformation gradient. Namely, you can "paint" a small square in the material, and then construct the deformation gradient by assembling the deformed material vectors into columns of $[\underset{\approx}{\boldsymbol{F}}]$. Now we will provide the *quantitative* definition of the deformation gradient. The deformation gradient is defined such that an infinitesimal material line segment $d\underset{\sim}{\boldsymbol{X}}$ deforms into a new infinitesimal line segment $d\underset{\sim}{\boldsymbol{x}}$ so that

$$d\underset{\sim}{\boldsymbol{x}} = \underset{\approx}{\boldsymbol{F}} \bullet d\underset{\sim}{\boldsymbol{X}} \tag{2.35}$$

or, by the chain rule,

$$\underset{\approx}{\boldsymbol{F}} = \frac{d\underset{\sim}{\boldsymbol{x}}}{d\underset{\sim}{\boldsymbol{X}}} \tag{2.36}$$

This conclusion holds because (recall) the deformation can be described through a mapping:

$$\underset{\sim}{\boldsymbol{x}} = \chi(\underset{\sim}{\boldsymbol{X}}) \tag{2.37}$$

In a rectangular Cartesian coordinate system, the components of the deformation gradient tensor are

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{2.38}$$

In matrix form,

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{2.39}$$

Examples may be found starting on page 38.

Homogenous deformations

We have described the deformation gradient tensor in the context of *inhomogenous* deformations. Referring to the definition in Eq. (2.38), note that the deformation gradient tensor itself generally varies with position. Consequently, the deformation of one infinitesimal material cubes generally will be different from that of another cube located elsewhere in the body.

A **homogenous** deformation is a very *special kind* of deformation in which the deformation gradient has the same value everywhere in the body. In this case, Eq. (2.36) can be integrated to give the mapping function,

$$\underset{\sim}{\boldsymbol{x}} = \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{X}} + \underset{\sim}{\boldsymbol{C}}, \tag{2.40}$$

where $\underset{\sim}{\boldsymbol{C}}$ is a constant of integration representing material translation. For homogenous deformations, straight lines deform to straight lines, planes deform to planes, cubes (no matter how large) deform to parallelepipeds, and spheres deform to ellipsoids. Contrast this with an *inhomogenous* deformation such as the one illustrated in Fig. 1.1, where originally straight grid lines become curved and only *infinitesimal* cubes deform to parallelepipeds. These distinctions are illustrated in Fig. 2.7.

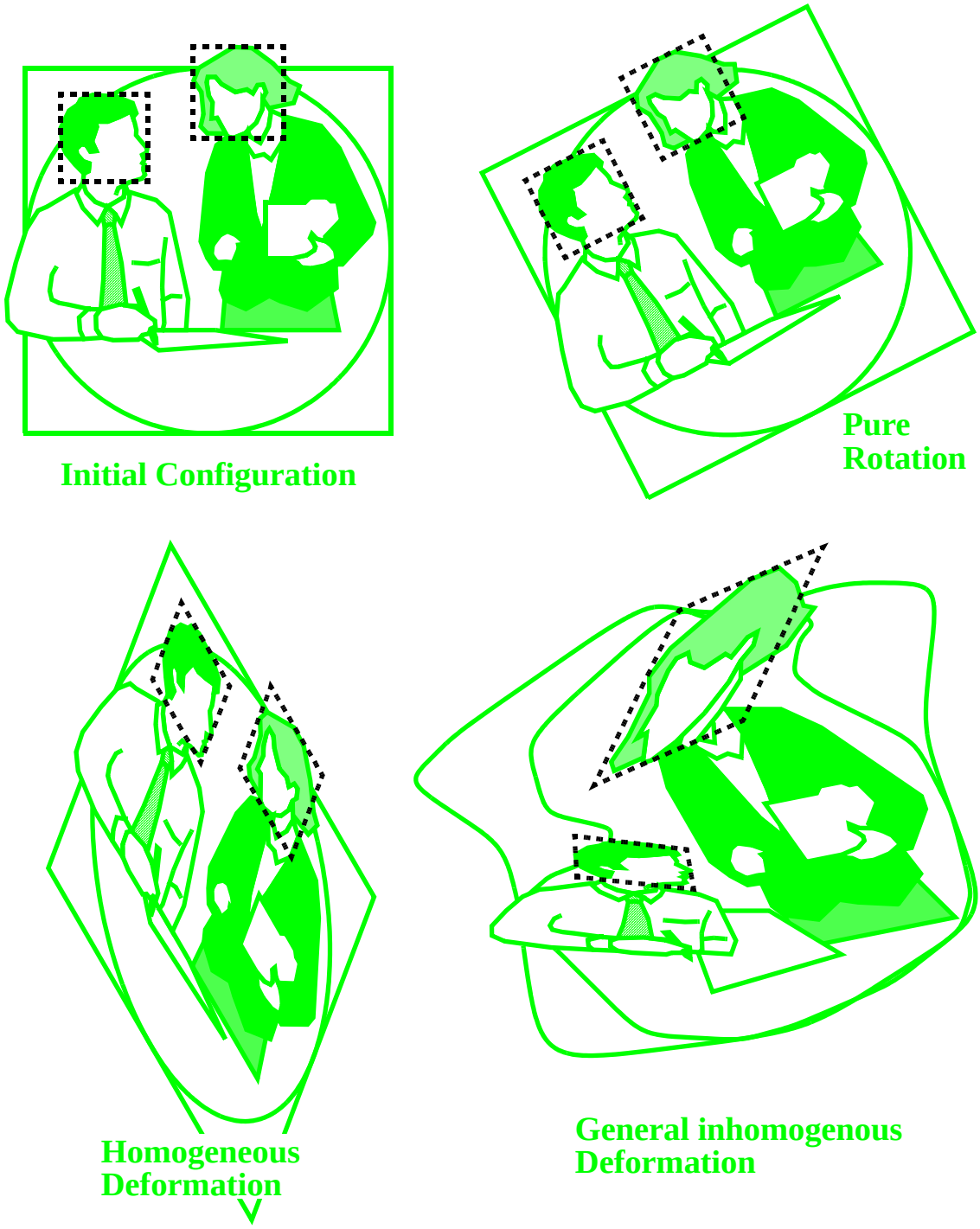


Figure 2.7. Distinctions between pure rotation, homogenous deformation and inhomogenous deformation.

Constructing the deformation gradient using dyads

In the previous example, we considered only the deformation of material fibers that were initially aligned with the lab basis. This is the easiest way to construct the deformation gradient tensor. If however, you start with three linearly independent material fibers $\{\underset{\sim}{M}_1, \underset{\sim}{M}_2, \underset{\sim}{M}_3\}$ (this time not necessarily of unit length) in the undeformed direction, then the procedure for constructing the deformation gradient must be modified. These concepts are useful in experimental diagnostics. For example, application to strain rosettes is discussed at the end of this section.

Method 1: You can assemble the *lab* components of the $\{\underset{\sim}{M}_1, \underset{\sim}{M}_2, \underset{\sim}{M}_3\}$ vectors into columns of a matrix $\underset{\approx}{f}_o$ so that

$$\underset{\sim}{M}_k = \underset{\approx}{f}_o \bullet \underset{\sim}{E}_k . \tag{2.41}$$

If these vectors become $\{\underset{\sim}{m}_1, \underset{\sim}{m}_2, \underset{\sim}{m}_3\}$ in the deformed configuration, then you can likewise assemble their *lab* components into a second matrix $\underset{\approx}{f}$ so that

$$\underset{\sim}{m}_k = \underset{\approx}{f} \bullet \underset{\sim}{E}_k . \tag{2.42}$$

We seek the deformation gradient tensor that connects the initial and final configuration directly. In other words, we seek a tensor $\underset{\approx}{F}$ such that

$$\underset{\sim}{m}_k = \underset{\approx}{F} \bullet \underset{\sim}{M}_k . \tag{2.43}$$

Solving Eq. (2.41) for $\underset{\sim}{E}_k$ gives $\underset{\sim}{E}_k = \underset{\approx}{f}_o^{-1} \bullet \underset{\sim}{M}_k$. Substituting this result into Eq. (2.42) gives

$$\underset{\sim}{m}_k = \underset{\approx}{f} \bullet \underset{\approx}{f}_o^{-1} \bullet \underset{\sim}{M}_k \tag{2.44}$$

Comparing Eq. (2.43) with Eq. (2.44) tells us that

$$\boxed{\underset{\approx}{F} = \underset{\approx}{f} \bullet \underset{\approx}{f}_o^{-1}} \tag{2.45}$$

Method 2: Another way to achieve the same result is to employ dyads and dual bases. Again, let $\{\underset{\sim}{M}_1, \underset{\sim}{M}_2, \underset{\sim}{M}_3\}$ denote three linearly independent infinitesimal material vectors in the undeformed 3D configuration. Let $\{\underset{\sim}{\mu}_1, \underset{\sim}{\mu}_2, \underset{\sim}{\mu}_3\}$ be the associated dual (or "contravariant") vectors. That is,

$$\underset{\sim}{\mu}_1 = \frac{\underset{\sim}{M}_2 \times \underset{\sim}{M}_3}{M_o}, \qquad \underset{\sim}{\mu}_2 = \frac{\underset{\sim}{M}_3 \times \underset{\sim}{M}_1}{M_o}, \qquad \underset{\sim}{\mu}_3 = \frac{\underset{\sim}{M}_1 \times \underset{\sim}{M}_2}{M_o},$$
$$\text{where } M_o \equiv (\underset{\sim}{M}_1 \times \underset{\sim}{M}_2) \bullet \underset{\sim}{M}_3 \tag{2.46}$$

If the material line segments $\{\underset{\sim}{M}_1, \underset{\sim}{M}_2, \underset{\sim}{M}_3\}$ deform to become new line segments $\{\underset{\sim}{m}_1, \underset{\sim}{m}_2, \underset{\sim}{m}_3\}$, then the deformation gradient tensor may be constructed by

$$\underset{\sim}{\boldsymbol{F}} = \sum_{k=1}^{3} \underset{\sim}{\boldsymbol{m}}_k \otimes \underset{\sim}{\mu}_k \tag{2.47}$$

Here, the symbol $\otimes$ denotes dyadic multiplication. For any two vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$, the dyad, $\underset{\sim}{\boldsymbol{a}} \otimes \underset{\sim}{\boldsymbol{b}}$, is a second order tensor whose ij component equals $a_i b_j$.

Eq. (2.45) and Eq. (2.47) will both give the same result.

Example: Suppose that a mounting bracket for the landing gear on an airplane was photographed during the most recent inspection of the plane. Following an unusually rough landing, a new photograph revealed that the bracket had undergone some severe deformation as shown in Fig. 2.8. We will use the lines formed by the "ACME landers" company logo to determine the deformation gradient for the bracket. For the purpose of this example, we will presume that the bracket underwent homogenous deformation. Of course, in real life, the deformation would likely be inhomogenous and the technique illustrated below would have to be applied using smaller line segments (such pairs of intersecting microscopic scratches) to find the deformation at various points in the bracket.

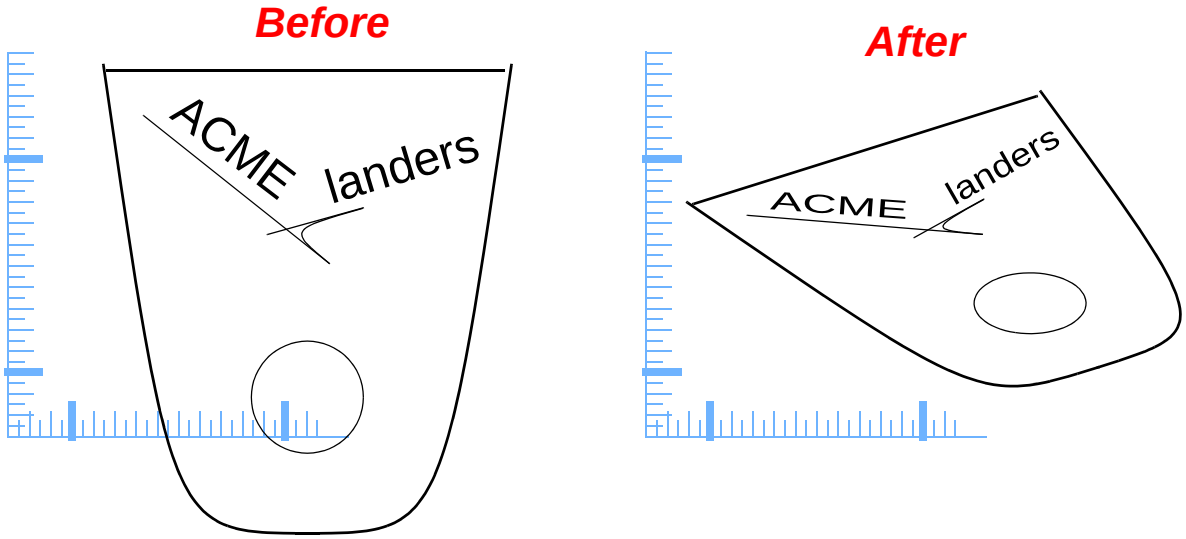


Figure 2.8. Airplane landing gear mounting bracket, before and after deformation

Introducing rulers as shown, the straight lines forming the company logo can be regarded as vectors. The line below the word "ACME" will be taken to be $\underset{\sim}{\boldsymbol{M}}_1$, pointing up and to the left; the line below "landers" will be $\underset{\sim}{\boldsymbol{M}}_2$, pointing up and to the right. Before deformation, the rulers show these vectors to have lab components as follows:

$$\underset{\sim}{\boldsymbol{M}}_1 = -0.875\underset{\sim}{\boldsymbol{E}}_1 + 0.7\underset{\sim}{\boldsymbol{E}}_2 \tag{2.48}$$

$$\underset{\sim}{\boldsymbol{M}}_2 = 0.45\underset{\sim}{\boldsymbol{E}}_1 + 0.125\underset{\sim}{\boldsymbol{E}}_2 \tag{2.49}$$

After deformation, the logo vectors deformed into

$$\underset{\sim}{\boldsymbol{m}}_1 = -1.1\underset{\sim}{\boldsymbol{E}}_1 + 0.1\underset{\sim}{\boldsymbol{E}}_2 \tag{2.50}$$

$$\underset{\sim}{\boldsymbol{m}}_2 = 0.325\underset{\sim}{\boldsymbol{E}}_1 + 0.175\underset{\sim}{\boldsymbol{E}}_2 \tag{2.51}$$

We will presume that the bracket did not deform at all out of plane so that

$$\underset{\sim}{\boldsymbol{M}}_3 = \underset{\sim}{\boldsymbol{E}}_3 \text{ and } \underset{\sim}{\boldsymbol{m}}_3 = \underset{\sim}{\boldsymbol{E}}_3 \tag{2.52}$$

Using method #1 for finding the deformation gradient tensor, the $\underset{\approx o}{\boldsymbol{f}}$ is constructed by assembling the lab components of the undeformed vectors into columns:

$$[\underset{\approx o}{\boldsymbol{f}}] = \begin{bmatrix} -0.875 & 0.45 & 0 \\ 0.7 & 0.125 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.53}$$

Similarly assembling the deformed vectors gives the $\underset{\approx}{\boldsymbol{f}}$ tensor as

$$[\underset{\approx}{\boldsymbol{f}}] = \begin{bmatrix} -1.1 & 0.325 & 0 \\ 0.1 & 0.175 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.54}$$

Applying Eq. (2.45) gives

$$[\underset{\approx}{\boldsymbol{F}}] = [\underset{\approx}{\boldsymbol{f}}][\underset{\approx o}{\boldsymbol{f}}]^{-1} = \begin{bmatrix} -1.1 & 0.325 & 0 \\ 0.1 & 0.175 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.875 & 0.45 & 0 \\ 0.7 & 0.125 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1.1 & 0.325 & 0 \\ 0.1 & 0.175 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.2945 & 1.060 & 0 \\ 1.6495 & 2.062 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.860 & -0.496 & 0 \\ 0.259 & 0.467 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.55}$$

Solving the same problem using method #2, we must first construct the dual vectors associated with $\{\underset{\sim}{\boldsymbol{M}}_1, \underset{\sim}{\boldsymbol{M}}_2, \underset{\sim}{\boldsymbol{M}}_3\}$. To start, note that

$$\underset{\sim}{\boldsymbol{M}}_1 \times \underset{\sim}{\boldsymbol{M}}_2 = -0.4244\underset{\sim}{\boldsymbol{E}}_3 \tag{2.56}$$

$$\underset{\sim}{\boldsymbol{M}}_2 \times \underset{\sim}{\boldsymbol{M}}_3 = 0.125\underset{\sim}{\boldsymbol{E}}_1 - 0.45\underset{\sim}{\boldsymbol{E}}_2 \tag{2.57}$$

$$\underset{\sim}{\boldsymbol{M}}_3 \times \underset{\sim}{\boldsymbol{M}}_1 = -0.7\underset{\sim}{\boldsymbol{E}}_1 - 0.875\underset{\sim}{\boldsymbol{E}}_2 \tag{2.58}$$

Therefore, referring to Eq. (2.46),

$$M_o \equiv (\underset{\sim}{\boldsymbol{M}}_1 \times \underset{\sim}{\boldsymbol{M}}_2) \bullet \underset{\sim}{\boldsymbol{M}}_3 = -0.4244 \tag{2.59}$$

$$\begin{aligned} \underset{\sim}{\mu}_1 &= -0.29455\underset{\sim}{\boldsymbol{E}}_1 + 1.0604\underset{\sim}{\boldsymbol{E}}_2 \\ \underset{\sim}{\mu}_2 &= 1.6495\underset{\sim}{\boldsymbol{E}}_1 + 2.06186\underset{\sim}{\boldsymbol{E}}_2 \\ \underset{\sim}{\mu}_3 &= \underset{\sim}{\boldsymbol{E}}_3, \end{aligned} \tag{2.60}$$

Recall that the deformed vectors were measured to be

$$\underset{\sim}{\boldsymbol{m}}_1 = -1.1\underset{\sim}{\boldsymbol{E}}_1 + 0.1\underset{\sim}{\boldsymbol{E}}_2 \tag{2.61}$$

$$\underset{\sim}{\boldsymbol{m}}_2 = 0.325\underset{\sim}{\boldsymbol{E}}_1 + 0.175\underset{\sim}{\boldsymbol{E}}_2 \tag{2.62}$$

Referring to Eq. (2.47), we need to compute the dyads, $\underset{\sim}{\boldsymbol{m}}_1 \otimes \underset{\sim}{\mu}_1$, $\underset{\sim}{\boldsymbol{m}}_2 \otimes \underset{\sim}{\mu}_2$, and $\underset{\sim}{\boldsymbol{m}}_3 \otimes \underset{\sim}{\mu}_3$:

$$\underset{\sim}{\boldsymbol{m}}_1 \otimes \underset{\sim}{\mu}_1 = \begin{bmatrix} -1.1 \\ 0.1 \\ 0 \end{bmatrix} \begin{bmatrix} -0.29455 & 1.0604 & 0 \end{bmatrix} = \begin{bmatrix} 0.324005 & -1.16644 & 0 \\ -0.029455 & 0.10604 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.63}$$

$$\underset{\sim}{\boldsymbol{m}}_2 \otimes \underset{\sim}{\mu}_2 = \begin{bmatrix} 0.325 \\ 0.175 \\ 0 \end{bmatrix} \begin{bmatrix} 1.6495 & 2.06186 & 0 \end{bmatrix} = \begin{bmatrix} 0.5360875 & 0.6701045 & 0 \\ 0.2886625 & 0.3608255 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.64}$$

$$[\underset{\sim}{\boldsymbol{m}}_2 \otimes \underset{\sim}{\mu}_2] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.65}$$

Now, applying Eq. (2.47) — adding up the above tensors — gives

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} 0.860 & -0.496 & 0 \\ 0.259 & 0.467 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.66}$$

which agrees with the result obtained using method #1. Obviously, for this particular problem, method #1 is superior in its computational simplicity. The fundamental concepts associated with method #2 are nonetheless important to understand and often easier to work with in theoretical discussions.

Strain rosette example: a strain rosette orients three strain gages at equal ($120°$) angles. To ensure measurement at a single point, the gages are centered at the same location; however, to analyze the rosette, you can imagine that the gages are actually placed along edges of an equilateral triangle and then presume that the material in the triangle deforms homogeneously by the same amount as the original central point. Strain gages return information about length changes but not orientation changes. In principle, if you could accurately measure the orientation changes of the first two gages, then the third one would not be necessary. The redundant third strain gage directly measures the *relative* orientation change between the first two gages. Thus, strain rosettes really measure only *stretch*, they do not characterize rotation. The full deformation gradient (which includes rotation) can always be constructed by first analyzing the rosette with the "fiducial" assumption that the first gage does not rotate. This will produce a fiducial deformation gradient and the actual deformation gradient can be constructed by then superimposing the actual rotation of the first gage onto the fiducial deformation.

3. Polar decomposition

A **rotation** is a special kind of deformation in which material vectors permissibly change orientation, but they don't change length. In this case, it can be shown [see gobag rotation document] that the associated deformation gradient will be *orthogonal* (its inverse will equal its transpose). Furthermore, since element inversions are prohibited, rotations will be *proper* orthogonal (determinant will equal $+1$) Rotations are discussed in great detail in the gobag "Rotation" document.

A **stretch** is a completely different special kind of deformation in which there exist three material vector orientations in the 3D initial configuration that will change in length, but not in orientation. In this case, the deformation gradient tensor will be both symmetric *and positive definite*. The proviso about positive definiteness is important. If a deformation gradient tensor is symmetric, that does *not* necessarily mean it is a stretch. For example, a pure rotation of $180°$ will result in a symmetric deformation gradient, but it's not a stretch because it possesses a negative eigenvalue. Being symmetric, a stretch is diagonal in its principal basis. The principal values, called principal stretches, equal the ratio of deformed to undeformed lengths of the three non-rotating material fibers. Material fibers that are *not* aligned with the principal directions of a stretch will change orientation, but for every fiber rotating one way, there will be another rotating in the opposite direction, making the net rotation of material fibers zero for a pure stretch deformation.

The polar decomposition theorem is a mathematical statement that the (local) deformation of a material element may be regarded as a combination of a pure stretch and a pure rotation. You can stretch a material and then rotate it or vice versa. Recall that order of application of sequentially applied deformations matters. Therefore, if you want to end up with the *same* deformation, then you must conclude that the stretch used in the "stretch-then-rotate" scenario would have to be different from the stretch used in the "rotate-then-stretch" scenario. You might also suspect that the rotations would also have to be different in the two scenarios. While it is true that the stretches are different, it can be proved that the rotation is the same in both cases. The polar decomposition states that there exists a rotation tensor $\underset{\approx}{\boldsymbol{R}}$, a "stretch-first" tensor $\underset{\approx}{\boldsymbol{U}}$ and a "stretch last" tensor $\underset{\approx}{\boldsymbol{V}}$ such that

Stretch then rotate: $\underset{\approx}{\boldsymbol{F}} = \underset{\approx}{\boldsymbol{R}} \bullet \underset{\approx}{\boldsymbol{U}}$ Note: $\underset{\approx}{\boldsymbol{U}}$ is called the "right stretch" (3.1)

Rotate then stretch: $\underset{\approx}{\boldsymbol{F}} = \underset{\approx}{\boldsymbol{V}} \bullet \underset{\approx}{\boldsymbol{R}}$ Note: $\underset{\approx}{\boldsymbol{V}}$ is called the "left stretch" (3.2)

Note that the Jacobian is $J = \det\underset{\approx}{\boldsymbol{F}} = \det\underset{\approx}{\boldsymbol{U}} = \det\underset{\approx}{\boldsymbol{V}}$. The following page summarizes key polar decomposition formulas (proof omitted).

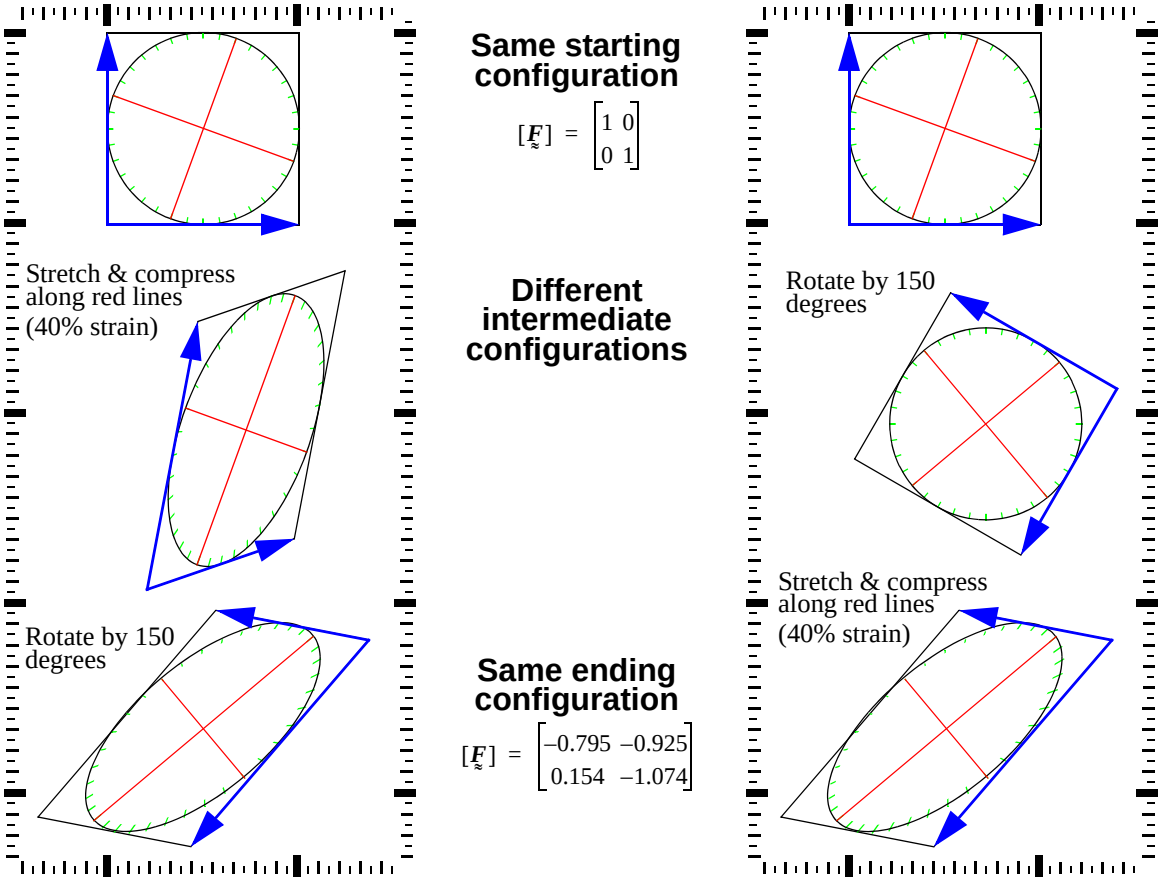
POLAR DECOMPOSITION

$\underset{\approx}{F} = \underset{\approx}{R} \bullet \underset{\approx}{U}$

First STRETCH
then ROTATE

$\underset{\approx}{F} = \underset{\approx}{V} \bullet \underset{\approx}{R}$

First ROTATE
then STRETCH



This figure shows a deformation for which principal Hencky (logarithmic) strains are $\pm 40\%$ and the material rotation is $150°$. This deformation can be accomplished via two stages: stretch and rotation. The polar decomposition theorem says that you will get the same final result regardless of whether you rotate first or rotate last. The key governing equations are:

$$\underset{\approx}{U} = (\underset{\approx}{F}^T \bullet \underset{\approx}{F})^{1/2} \qquad \underset{\approx}{V} = (\underset{\approx}{F} \bullet \underset{\approx}{F}^T)^{1/2} = \underset{\approx}{R} \bullet \underset{\approx}{U} \bullet \underset{\approx}{R}^T$$

$$\underset{\approx}{R} = \underset{\approx}{F} \bullet \underset{\approx}{U}^{-1} \qquad \underset{\approx}{R} = \underset{\approx}{V}^{-1} \bullet \underset{\approx}{F}$$

For 2D deformations, the rotation tensor can be directly computed by

$$[\underset{\approx}{R}] = \frac{1}{\sqrt{(F_{11}+F_{22})^2 + (F_{12}-F_{21})^2}} \begin{bmatrix} F_{11}+F_{22} & F_{12}-F_{21} \\ F_{21}-F_{12} & F_{22}+F_{11} \end{bmatrix}$$

Then $\underset{\approx}{U} = \underset{\approx}{R}^T \bullet \underset{\approx}{F}$ and $\underset{\approx}{V} = \underset{\approx}{F} \bullet \underset{\approx}{R}^T$

Recall that the deformation of an infinitesimal line element $d\underset{\sim}{\boldsymbol{X}}$ is given by

$$d\underset{\sim}{\boldsymbol{x}} = \underset{\approx}{\boldsymbol{F}} \bullet d\underset{\sim}{\boldsymbol{X}} \tag{3.3}$$

In terms of the polar rotation and stretch, this becomes

$$d\underset{\sim}{\boldsymbol{x}} = \underset{\approx}{\boldsymbol{R}} \bullet \underset{\approx}{\boldsymbol{U}} \bullet d\underset{\sim}{\boldsymbol{X}} \tag{3.4}$$

Broken down in this manner, we see that $\underset{\approx}{\boldsymbol{U}} \bullet d\underset{\sim}{\boldsymbol{X}}$ quantifies the stretching of the material fiber $d\underset{\sim}{\boldsymbol{X}}$, as well as the part of the fiber rotation that results strictly from the material *distortion* (i.e., its shape change only). Under the pure stretch, $\underset{\approx}{\boldsymbol{U}} \bullet d\underset{\sim}{\boldsymbol{X}}$, fibers that are aligned with the principal directions of $\underset{\approx}{\boldsymbol{U}}$ won't rotate. Fibers not aligned with the principal directions will rotate, but for any fiber that rotates one way, there will exist a different fiber rotating in the opposite direction. Consequently, the overall material rotation under a pure stretch will be zero. The total deformation is obtained after the pure stretch by the application of the rotation tensor $\underset{\approx}{\boldsymbol{R}}$, which rotates *all* fibers about the same axis by the same angle of rotation. Hence, $\underset{\approx}{\boldsymbol{R}}$ is really is an excellent measure of overall material rotation.

EXAMPLE: Consider the deformation in Eq. (2.66):

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} 0.860 & -0.496 & 0 \\ 0.259 & 0.467 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

This is a two-dimensional deformation, so the rotation tensor is found by

$$\begin{aligned} [\underset{\approx}{\boldsymbol{R}}] &= \frac{1}{\sqrt{(F_{11}+F_{22})^2 + (F_{12}-F_{21})^2}} \begin{bmatrix} F_{11}+F_{22} & F_{12}-F_{21} \\ F_{21}-F_{12} & F_{22}+F_{11} \end{bmatrix} \\ &= \frac{1}{\sqrt{(1.327)^2 + (-0.755)^2}} \begin{bmatrix} 1.327 & -0.755 \\ 0.755 & 1.327 \end{bmatrix} \\ &= \begin{bmatrix} 0.8692 & -0.4945 \\ 0.4945 & 0.8692 \end{bmatrix} \end{aligned} \tag{3.6}$$

Since, for 2D deformations, $R_{11} = \cos\alpha$ and $R_{21} = \sin\alpha$, where α is the rotation angle, we note that $\alpha = 30°$.

The reference stretch is

$$[\underset{\sim}{\boldsymbol{U}}] = [\underset{\sim}{\boldsymbol{R}}]^T[\underset{\sim}{\boldsymbol{F}}] = \begin{bmatrix} 0.8692 & 0.4945 \\ -0.4945 & 0.8692 \end{bmatrix} \begin{bmatrix} 0.860 & -0.496 \\ 0.259 & 0.467 \end{bmatrix} = \begin{bmatrix} 0.876 & -0.200 \\ -0.200 & 0.651 \end{bmatrix} \tag{3.7}$$

The spatial stretch is

$$[\underset{\sim}{\boldsymbol{V}}] = [\underset{\sim}{\boldsymbol{F}}][\underset{\sim}{\boldsymbol{R}}]^T = \begin{bmatrix} 0.860 & -0.496 \\ 0.259 & 0.467 \end{bmatrix} \begin{bmatrix} 0.8692 & 0.4945 \\ -0.4945 & 0.8692 \end{bmatrix} = \begin{bmatrix} 0.993 & -0.006 \\ -0.006 & 0.534 \end{bmatrix} \tag{3.8}$$

Note that the off-diagonals (V_{12} and V_{21}) are nearly equal to zero, which means the principal directions of the spatial stretch just happened to turn out to coincide with the lab basis for this particular example. Also note that the 11 component of the stretch is nearly equal to 1. For this particular problem, if it weren't for computational round-off, V_{12} and V_{21} have turned out to be identically zero and V_{11} would have been identically equal to 1. We are able to make this assertion because we know how we created Fig. 2.8 (which corresponds to this deformation) in the first place. That figure was created by using the computer's drawing tools to first rotate the undeformed configuration by an angle $\alpha = 30°$ and *then* to reduce the height of the rotated drawing by approximately 50% (note that V_{22} is nearly equal to 50% of unity). Since the drawing was obtained by first rotating and then stretching, we know that the rotation angle computed from the polar decomposition had to come out equal to our initial rotation angle, and the *spatial* stretch had to come out to be a 50% vertical compression, as it has! We could have alternatively constructed the drawing by using a stretch then rotate sequence involving a 50% compression of the initial configuration along a line oriented at $-30°$ followed by a rotation of $+30°$. In general, the principal directions of stretch are neither coincident with the lab directions nor related in any way to the rotation angle.

Incidentally, upgrading the above stretches and rotations to 3D gives

$$[\underset{\approx}{\boldsymbol{R}}] = \begin{bmatrix} 0.8692 & -0.4945 & 0 \\ 0.4945 & 0.8692 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

$$[\underset{\approx}{\boldsymbol{U}}] = \begin{bmatrix} 0.876 & -0.200 & 0 \\ -0.200 & 0.651 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

$$[\underset{\approx}{\boldsymbol{V}}] = \begin{bmatrix} 0.993 & -0.006 & 0 \\ -0.006 & 0.534 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.11}$$

4. Strain

Linear strain measures

The deformation gradient is truly the best measure of deformation. It is especially attractive because it contains information about *both* stretch and material rotation. For one-dimensional homogenous *uniaxial* deformations, the axial strain is often defined as

$$\varepsilon = \frac{L - L_o}{L_o} = \lambda - 1, \text{ where } \boxed{\lambda = \frac{L}{L_o}} \tag{4.1}$$

The parameter λ is called the "axial stretch."

Axial strain is often defined alternatively as

$$\varepsilon = \frac{L - L_o}{L} = -[\lambda^{-1} - 1] \tag{4.2}$$

Many advanced materials models define the strain as

$$\varepsilon = \ln\lambda \tag{4.3}$$

Note that all you really need to describe the amount of deformation is the stretch λ. All of these strains definitions are of the general "Seth-Hill" form,*

$$\boxed{\varepsilon = \frac{1}{k}[\lambda^k - 1]} \tag{4.4}$$

The parameter k characterizes the strain definition. By appropriate choices for k, any of the above strain definitions can be achieved. For example, Eq. (4.1) is obtained by choosing k=1, while Eq. (4.2) corresponds to k=–1. The log strain in Eq. (4.3), also known as Hencky strain, is obtained in the limit as $k \to 0$. All of these various strain definitions are equivalent to first order when length changes are small (i.e., when λ is close to 1). That's because a two-term Taylor series expansion of Eq. (4.4) centered at λ=1 gives $\varepsilon \sim \lambda - 1$, regardless of the value of k.

The variation of strain with stretch is shown in Fig. 4.1 for various choices of the Seth-Hill parameter k. All of the strain measures are negative in compression $(\lambda < 1)$ and positive in tension $(\lambda > 1)$. For positive choices of k, extreme compression toward the limit as $\lambda \to 0$ results in the strain measure approaching a *finite* limit. Physically, one would expect that an infinite load would be required in order to compress a material down to nothing. Consequently, if a choice $k > 0$ were to be used in a *linear* stress-strain relationship, $\sigma = E\varepsilon$, this behavior would imply that complete compression (a physical impossibility) could be obtained via a finite load. Such a result is nonphysical and therefore a nonlinear constitutive model *must* be used when using $k > 0$ in a highly compressed material. Otherwise numerical solutions will exhibit compression instabilities. When choosing a *negative* Seth-Hill parameter $(k < 0)$, the opposite happens: positive choices for k become unstable if a linear stress-strain relation is used in a highly tensile (large stretch) application.† The logarithmic strain $(k \to 0)$ is the *only* strain choice that avoids both of these (compression and tension) pitfalls. This

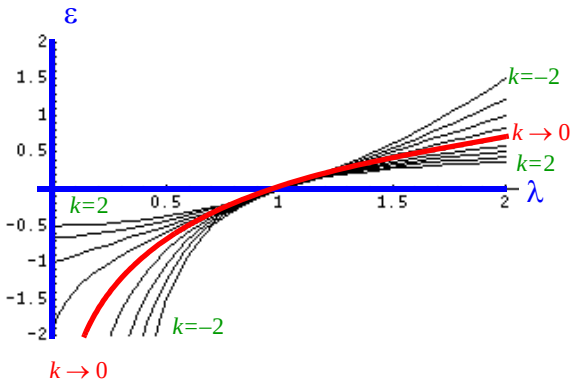


Figure 4.1. The Seth-Hill family of strain measures.

* Referred to as "Erikson-Doyle" in the rheology literature.

does not mean that a material's stress-strain relationship will necessarily be linear with respect to a logarithmic strain measure; it only means that a first-order Taylor series expansion of the actual (generally nonlinear) stress-strain function is likely to be more accurate than other strain measures. Other strain measures can be used, and the same level of accuracy can be obtained if more terms in a Taylor series expansion are employed.

Recall that a stretch is the ratio of deformed to undeformed lengths. For general deformations, recall that we have already shown that the Jacobian ($J = \det\underset{\approx}{\boldsymbol{F}} = \det\underset{\approx}{\boldsymbol{U}} = \det\underset{\approx}{\boldsymbol{V}}$) equals the ratio of deformed to undeformed volumes. Thus, J is the volumetric analog of the linear stretch λ and we may define *volume* strains in a manner consistent with the linear strain definition given in Eq. (4.4). Namely, the "consistent" volumetric strain is defined

$$\varepsilon_v = \frac{1}{k}[J^k - 1] \tag{4.5}$$

This volumetric strain is well posed for any 3D deformation. Soon, we will define strain tensors and then prove that the consistent volumetric strain is *not* equal to the trace of the strain tensor except when $k=0$ (i.e., log strain).

Strain tensors

For 3D deformations, linear strain measures can still be defined in terms of the ratio of the deformed to undeformed lengths of an individual material fiber. However, for 3D, differently oriented fibers will experience different amounts of stretch. Furthermore, two material fibers can change their relative orientations under 3D deformations, so we need to quantify that phenomenon as well. We have already emphasized that the deformation gradient tensor already does an excellent job in this respect. The deformation gradient is, by far, the superior measure of material deformation. Nonetheless, a lot of people are more comfortable with using *strain* instead of the deformation gradient, so we will describe now how to compute strain from a deformation gradient tensor. Keep in mind that strain tensors lose information (about rotation); you can construct any of the strain tensors if you have the deformation gradient, but you cannot do the reverse.

Recall that the 1D definition of strain given in the previous section was phrased in terms of the ratio of deformed to undeformed lengths. We can extend this concept to 3D by making use of the polar decomposition theorem. The stretch *tensor* from the polar decomposition is symmetric and positive definite. Consequently, it is diagonal in its principal basis. In the absence of material rotation (i.e., if $\underset{\approx}{\boldsymbol{R}} = \underset{\approx}{\boldsymbol{I}}$) any material fiber oriented in one the i^{th} principal stretch directions will not change orientation, but it will change length by a factor equal to the eigenvalue, λ_i. This suggests that we can use the Seth-Hill generalized strain definition in each of these directions, and then assemble the resulting linear strains into a tensor that is diagonal with respect to the stretch directions. When

† This is less likely to be troublesome in practice because usually some other inelastic failure mechanism (e.g. fracture) usually initiates well before the unstable tensile regime is reached.

there *is* material rotation (i.e., if $\underset{\approx}{\boldsymbol{R}} \neq \underset{\approx}{\boldsymbol{I}}$) then two strain definitions are possible, one that uses the reference stretch and one that uses the spatial stretch. That is, to define a 3D generalized strain tensor, we can "upgrade" the Seth-Hill strain measure in Eq. (4.4) to either a *reference* strain,

$$\underset{\approx}{\bar{\varepsilon}} = \frac{1}{k}[\underset{\approx}{\boldsymbol{U}}^k - \underset{\approx}{\boldsymbol{I}}] \tag{4.6}$$

or a *spatial* strain,

$$\underset{\approx}{\varepsilon} = \frac{1}{k}[\underset{\approx}{\boldsymbol{V}}^k - \underset{\approx}{\boldsymbol{I}}] \tag{4.7}$$

Of course, if $k=0$ is used, the expressions become, in the limit,

$$\underset{\approx}{\bar{\varepsilon}} = \ln(\underset{\approx}{\boldsymbol{U}}) \qquad \text{and} \qquad \underset{\approx}{\varepsilon} = \ln(\underset{\approx}{\boldsymbol{V}}) \tag{4.8}$$

In general, for non-even powers of k,* the only way to compute $\underset{\approx}{\boldsymbol{U}}^k$ [or to compute $\ln(\underset{\approx}{\boldsymbol{U}})$] requires expressing $\underset{\approx}{\boldsymbol{U}}$ in its principal basis, raising each principal stretch to the k^{th} power [or taking logs of the principal stretches if log strain is desired], and then transforming back to the lab basis. Specifically if $[U]$ is the matrix of lab components of $\underset{\approx}{\boldsymbol{U}}$ and if a matrix $[Q]$ is constructed to contain the lab components of the orthonormalized principal stretch basis, then

$$[U] = [Q]^T[\Lambda][Q] \tag{4.9}$$

where $[\Lambda]$ is a diagonal matrix containing the principal stretches associated with the principal basis. The components of the general Seth-Hill strain are found by

$$[\underset{\approx}{\bar{\varepsilon}}] = \frac{1}{k}([Q][\Lambda^k][Q]^T - [I]) \tag{4.10}$$

Similarly, the components of the logarithmic strain with respect to the lab basis are found by

$$[\underset{\approx}{\bar{\varepsilon}}] = [Q][\ln\Lambda][Q]^T \tag{4.11}$$

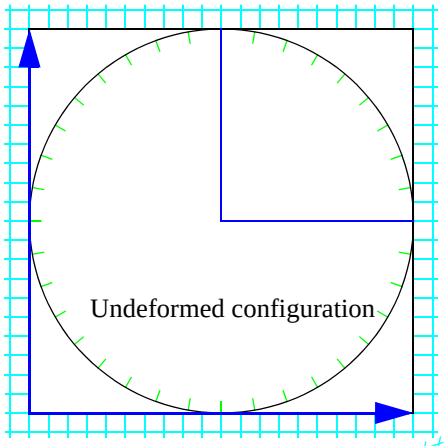
where $[\ln\Lambda]$ is a diagonal matrix containing the natural logs of the principal stretches. For the logarithmic strain, note that

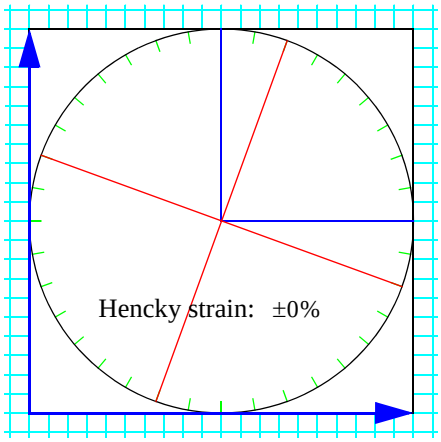
$$\text{tr}\underset{\approx}{\bar{\varepsilon}} = \ln(\lambda_1) + \ln(\lambda_2) + \ln(\lambda_3) = \ln(\lambda_1\lambda_2\lambda_3) = \ln J = \varepsilon_v \tag{4.12}$$

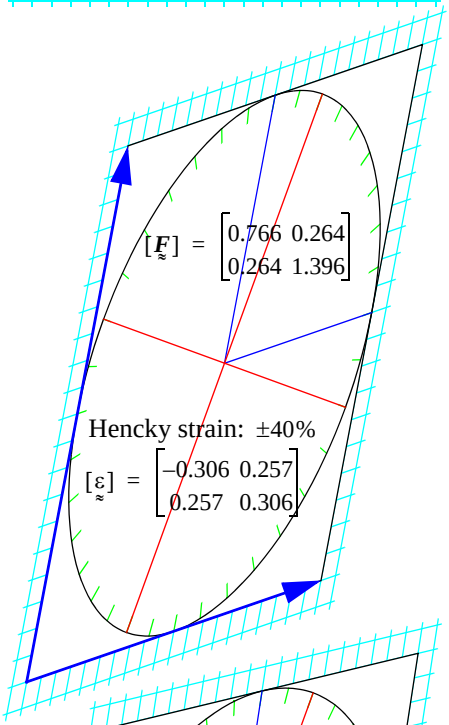
The logarithmic strain is the only large-deformation strain measure whose trace equals the consistent volumetric strain. Some sketches of pure stretch deformations and their associated logarithmic principal strains are given on the following page.

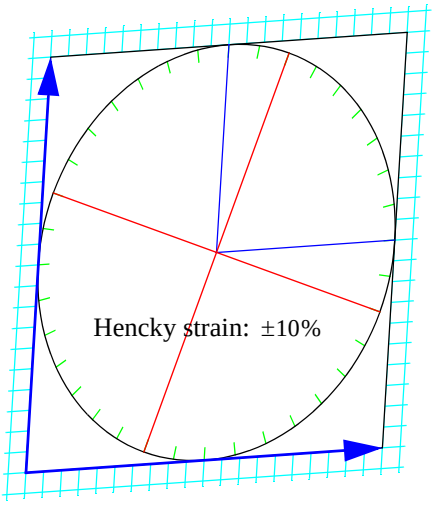
* We will explain later why an eigenvalue decomposition is not necessary for even Seth-Hill parameters. That's the key reason why even powers are used at all -- they are simpler computationally.

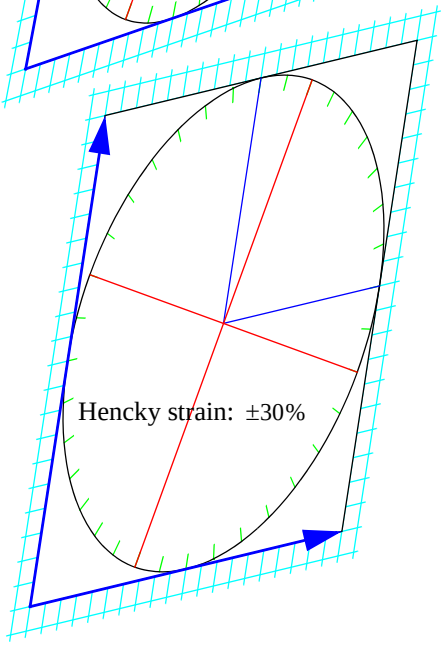
PURE STRETCH

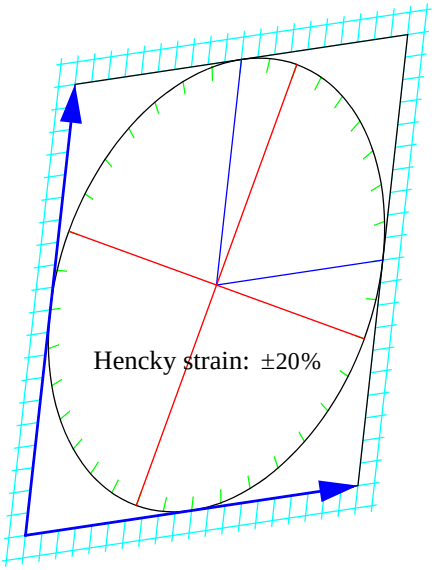




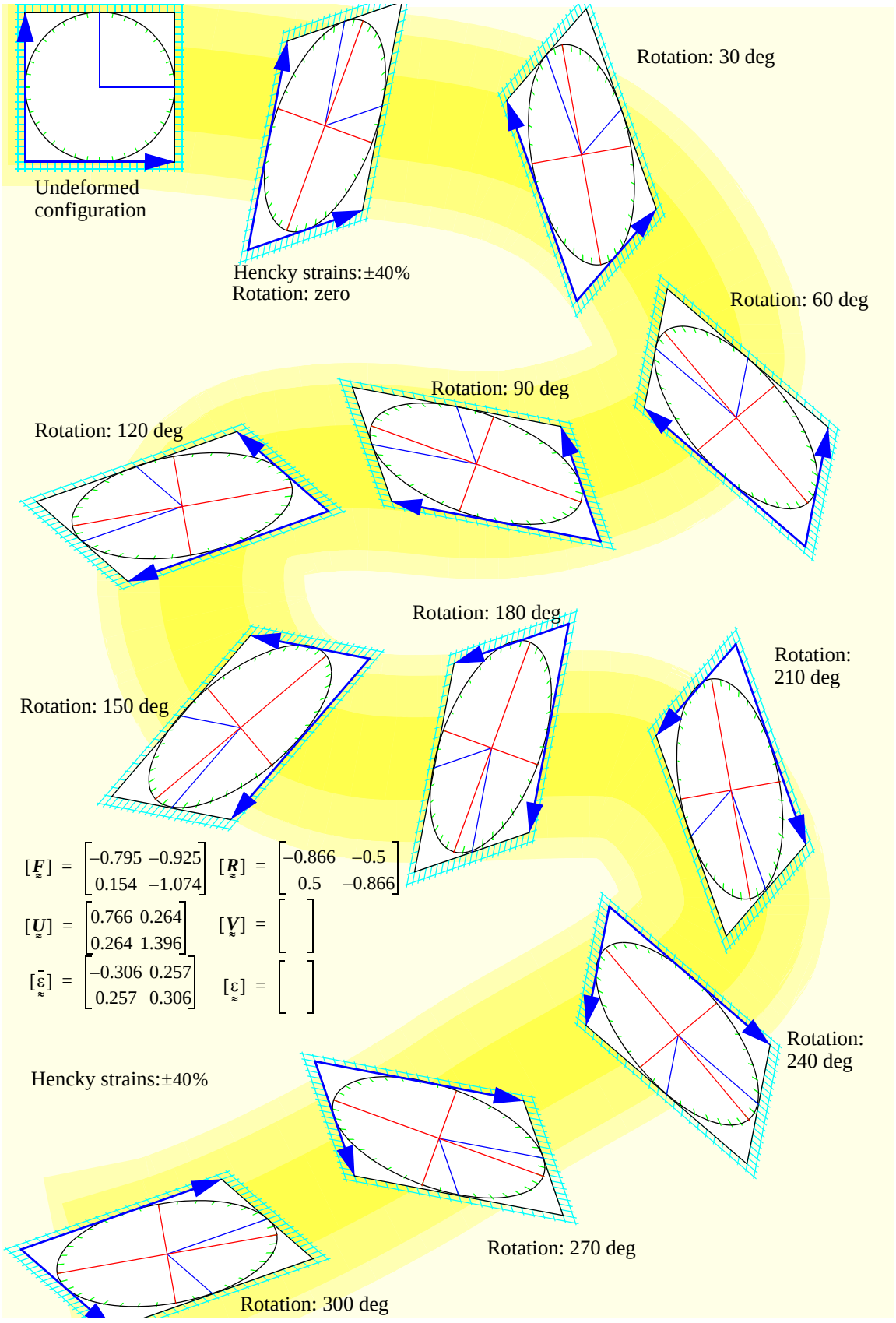








PURE STRETCH FOLLOWED BY ROTATIONS



$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} -0.795 & -0.925 \\ 0.154 & -1.074 \end{bmatrix} \quad [\underset{\approx}{\boldsymbol{R}}] = \begin{bmatrix} -0.866 & -0.5 \\ 0.5 & -0.866 \end{bmatrix}$$

$$[\underset{\approx}{\boldsymbol{U}}] = \begin{bmatrix} 0.766 & 0.264 \\ 0.264 & 1.396 \end{bmatrix} \quad [\underset{\approx}{\boldsymbol{V}}] = \begin{bmatrix} \\ \end{bmatrix}$$

$$[\underset{\approx}{\bar{\varepsilon}}] = \begin{bmatrix} -0.306 & 0.257 \\ 0.257 & 0.306 \end{bmatrix} \quad [\underset{\approx}{\varepsilon}] = \begin{bmatrix} \\ \end{bmatrix}$$

For pure stretch deformations, there is no difference between $\bar{\varepsilon}$ and ε defined in Eqs. (4.6) and (4.7). However, whenever there is material rotation, the spatial and reference strains will differ from each other by

$$\underset{\approx}{\varepsilon} = \underset{\approx}{\boldsymbol{R}} \bullet \underset{\approx}{\bar{\varepsilon}} \bullet \underset{\approx}{\boldsymbol{R}}^T, \tag{4.13}$$

where $\underset{\approx}{\boldsymbol{R}}$ is the rotation from the polar decomposition theorem. This distinction (illustrated on the previous page) becomes of paramount importance when constructing material constitutive laws for anisotropic materials. Modeling an anisotropic material requires specification of the material orientation. If a constitutive model is constructed using the *spatial* strain, then that model must transform the initial material orientation into the new (rotated) orientation prior to applying the model. Additionally, the principle of material frame indifference demands that models that use *spatial* strain must introduce special "objective" rates when ever the constitutive model is applied in rate form. When using the *reference* strain, however, neither rotation of the material orientation nor special rates are required. The price paid for using reference strain measures is the spatial strain (which is what's usually available from structural mechanics host codes) has to be transformed back into the unrotated configuration prior to application of the constitutive model, and then the output of the model must be transformed back to the spatial configuration. Computationally, the reference strain option is more efficient and guaranteed to satisfy the principle of material frame indifference.

We already mentioned that an advantage of the logarithmic strain is that its trace equals the volumetric strain. Furthermore, the log strain tends to be the choice that makes the stress-strain function "most linear" in uniaxial tests. However, the logarithmic strain requires a polar decomposition in order to obtain a stretch tensor, and the polar decomposition is labor-intensive because it requires an eigenvalue analysis to take the square root of a tensor. Moreover, computing *rates* of the logarithmic strain is extraordinarily complicated because, in addition to the principal stretches being a function of time, the principal *directions* also change in time. For many people, these disadvantages far outweigh the advantages.

Recall (from the polar decomposition summary sheet) that $\underset{\approx}{\boldsymbol{U}} = (\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}})^{1/2}$. Consequently, the generalized Seth-Hill reference strain in Eq. (4.6) may be written

$$\underset{\approx}{\bar{\varepsilon}} = \frac{1}{k}[(\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}})^{k/2} - \underset{\approx}{\boldsymbol{I}}] \tag{4.14}$$

All of the disadvantages we cited for the logarithmic strain will apply whenever the Seth-Hill parameter not an *even* (and nonzero) integer. *Even* values for the parameter k are computationally attractive because there is no need to compute the square root of a tensor. In particular, the choice k=2 corresponds to the what is called the "Lagrangean strain" tensor. With this choice, the "square root" in Eq. (4.14) goes away, giving

$$\underset{\approx}{\boldsymbol{E}} = \frac{1}{2}[\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}} - \underset{\approx}{\boldsymbol{I}}] \tag{4.15}$$

The Lagrange strain is a *reference* strain. Consequently, it may be regarded as a strain that results from the pure reference (right) stretch prior to the rotation part of the deformation. Superimposing even more rotation will have no effect on the Lagrange strain. Anisotropic constitutive models that use the Lagrange strain in conjunction with a reference stress measure do not require rotation of the material orientation into the current state. These models will automatically satisfy the principle of material frame indifference without needing to introduce any special objective rates.

The Lagrange strain can be readily computed *directly* from the deformation gradient tensor (no polar decomposition is required), and its rate is also easily found by

$$\dot{\underset{\approx}{\boldsymbol{E}}} = \frac{1}{2}[\dot{\underset{\approx}{\boldsymbol{F}}}^T \bullet \underset{\approx}{\boldsymbol{F}} + \underset{\approx}{\boldsymbol{F}}^T \bullet \dot{\underset{\approx}{\boldsymbol{F}}}] \tag{4.16}$$

A fundamental identity from continuum mechanics states that the rate of the deformation gradient is related to the material's spatial velocity gradient, $\underset{\approx}{\boldsymbol{L}} \equiv \partial \underset{\sim}{\boldsymbol{v}} / \partial \underset{\sim}{\boldsymbol{x}}$, by

$$\dot{\underset{\approx}{\boldsymbol{F}}} = \underset{\approx}{\boldsymbol{L}} \bullet \underset{\approx}{\boldsymbol{F}} \tag{4.17}$$

Therefore, Eq. (4.16) can be written

$$\dot{\underset{\approx}{\boldsymbol{E}}} = \underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{D}} \bullet \underset{\approx}{\boldsymbol{F}}, \text{ where } \underset{\approx}{\boldsymbol{D}} \equiv \frac{1}{2}[\underset{\approx}{\boldsymbol{L}} + \underset{\approx}{\boldsymbol{L}}^T] \tag{4.18}$$

The tensor $\underset{\approx}{\boldsymbol{D}}$ is often called the "rate of deformation," which is an unfortunate and misleading name because, despite having some rate-like properties, it is not the proper rate of any tensor. By this, we mean that it is possible to construct closed deformation paths* for which the time integral of $\underset{\approx}{\boldsymbol{D}}$ is not zero. Frequently in mechanics, it is useful to introduce a general "unrotation" operation, denoted by an overbar, and defined for any *spatial* second-order tensor $\underset{\approx}{\boldsymbol{Y}}$ by

$$\overline{\underset{\approx}{\boldsymbol{Y}}} \equiv \underset{\approx}{\boldsymbol{R}}^T \bullet \underset{\approx}{\boldsymbol{Y}} \bullet \underset{\approx}{\boldsymbol{R}} \tag{4.19}$$

Using the overbar notation, Fig. 4.18 can be written

$$\dot{\underset{\approx}{\boldsymbol{E}}} = \overline{\underset{\approx}{\boldsymbol{V}}} \bullet \overline{\underset{\approx}{\boldsymbol{D}}} \bullet \overline{\underset{\approx}{\boldsymbol{V}}}, \tag{4.20}$$

where

$$\overline{\underset{\approx}{\boldsymbol{D}}} \equiv \underset{\approx}{\boldsymbol{R}}^T \bullet \underset{\approx}{\boldsymbol{D}} \bullet \underset{\approx}{\boldsymbol{R}} \tag{4.21}$$

and

$$\overline{\underset{\approx}{\boldsymbol{V}}} = \underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{R}} = \underset{\approx}{\boldsymbol{R}}^T \bullet \underset{\approx}{\boldsymbol{V}} \bullet \underset{\approx}{\boldsymbol{R}} = \underset{\approx}{\boldsymbol{U}} \tag{4.22}$$

* i.e., deformations that begin at one configuration, move through other configurations, and then return to the starting configuration.

Reference displacement gradients

The displacement gradient tensor is just the derivative of the displacement with respect to the reference position vector:

$$\underset{\approx}{\boldsymbol{H}} = \frac{d\underset{\sim}{\boldsymbol{u}}}{d\underset{\sim}{\boldsymbol{X}}}, \qquad \text{or, in RCS, } H_{ij} = \frac{\partial u_i}{\partial X_j} \tag{4.23}$$

Recalling that $\underset{\sim}{\boldsymbol{u}} = \underset{\sim}{\boldsymbol{x}} - \underset{\sim}{\boldsymbol{X}}$, note that

$$\underset{\approx}{\boldsymbol{H}} = \frac{d\underset{\sim}{\boldsymbol{u}}}{d\underset{\sim}{\boldsymbol{X}}} = \frac{d(\underset{\sim}{\boldsymbol{x}} - \underset{\sim}{\boldsymbol{X}})}{d\underset{\sim}{\boldsymbol{X}}} = \frac{d\underset{\sim}{\boldsymbol{x}}}{d\underset{\sim}{\boldsymbol{X}}} - \frac{d\underset{\sim}{\boldsymbol{X}}}{d\underset{\sim}{\boldsymbol{X}}} = \underset{\approx}{\boldsymbol{F}} - \underset{\approx}{\boldsymbol{I}} \tag{4.24}$$

Solving for $\underset{\approx}{\boldsymbol{F}}$ in Eq. (4.24) and substituting the result into Eq. (4.15) gives

$$\underset{\approx}{\overline{\boldsymbol{E}}} = \frac{1}{2}[\underset{\approx}{\boldsymbol{H}} + \underset{\approx}{\boldsymbol{H}}^T + \underset{\approx}{\boldsymbol{H}}^T \bullet \underset{\approx}{\boldsymbol{H}}] \tag{4.25}$$

This is eq. (3.19) in the book.

This is the formula one typically finds in an elasticity textbook, whereas continuum mechanics textbooks typically favor the more elegant and computationally simple Eq. (4.15).

A "wrong-headed" rotation measure

Many elasticity books define a tensor $\underset{\approx}{\Omega}$ by

$$\underset{\approx}{\Omega} = \frac{1}{2}[\underset{\approx}{\boldsymbol{H}} - \underset{\approx}{\boldsymbol{H}}^T - \underset{\approx}{\boldsymbol{H}}^T \bullet \underset{\approx}{\boldsymbol{H}}] \tag{4.26}$$

or

$$\underset{\approx}{\Omega} = \underset{\approx}{\boldsymbol{H}} - \underset{\approx}{\boldsymbol{E}} \tag{4.27}$$

This tensor is well-defined, but it is *not* a sensible or useful measure of material rotation, as claimed (by some) in the elasticity community. We will call this tensor the "wrong-headed rotation" (WHR) tensor. In terms of the deformation gradient, the WHR tensor may be written

$$\underset{\approx}{\Omega} = \underset{\approx}{\boldsymbol{F}} - \underset{\approx}{\boldsymbol{I}} - \frac{1}{2}[\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}} - \underset{\approx}{\boldsymbol{I}}] \tag{4.28}$$

In terms of the polar stretch and rotation tensors,

$$\underset{\approx}{\Omega} = \underset{\approx}{\boldsymbol{R}} \bullet \underset{\approx}{\boldsymbol{U}} - \underset{\approx}{I} - \frac{1}{2}[\underset{\approx}{\boldsymbol{U}}^2 - \underset{\approx}{\boldsymbol{I}}] \tag{4.29}$$

In the complete absence of material stretch (i.e., when $\underset{\approx}{\boldsymbol{U}} \approx \underset{\approx}{I}$) the tensor $\underset{\approx}{\Omega} + \underset{\approx}{\boldsymbol{I}}$ does indeed approximately equal the polar rotation $\underset{\approx}{\boldsymbol{R}}$. However, for finite stretches, such an interpretation does not apply and the WHR tensor is essentially useless. For finite stretch problems, forget about the WHR, and use the polar rotation tensor instead.

Under a pure stretch (i.e., non-rotational) deformation, (i.e., when $\underset{\approx}{\boldsymbol{F}} = \underset{\approx}{\boldsymbol{U}}$), the WHR tensor becomes $\underset{\approx}{\Omega} = \underset{\approx}{e} - \underset{\approx}{\boldsymbol{E}}$, where $\underset{\approx}{e}$ is the engineering $(k = 1)$ strain measure. Thus, since engineering strain is not equal to Lagrange strain, the WHR tensor will not be zero or the identity even under pure stretches. *It is a lousy measure of material rotation.*

Incidentally, noting that the last term in Eq. (4.28) is the Lagrange strain, the WHR tensor may be written

$$\underset{\approx}{\Omega} = \underset{\approx}{\boldsymbol{F}} - \underset{\approx}{\boldsymbol{I}} - \underset{\approx}{\overline{\boldsymbol{E}}} \tag{4.30}$$

Spatial strain measures

In the previous sections, we discussed the Lagrange strain tensor $\underset{\approx}{\overline{\boldsymbol{E}}}$, defined by

$$\underset{\approx}{\overline{\boldsymbol{E}}} = \frac{1}{2}[\underset{\approx}{\boldsymbol{U}}^2 - \underset{\approx}{\boldsymbol{I}}] \tag{4.31}$$

This strain measure is a reference strain measure because it is computed using the reference stretch tensor. It will not change if you superimpose extra rotation on the deformation. The Lagrange strain has a spatial counterpart, called the Signorini strain, defined

$$\underset{\approx}{\boldsymbol{E}} = \frac{1}{2}[\underset{\approx}{\boldsymbol{V}}^2 - \underset{\approx}{\boldsymbol{I}}] \tag{4.32}$$

The Signorini strain is related to the Lagrange strain by

$$\underset{\approx}{\boldsymbol{E}} = \underset{\approx}{\boldsymbol{R}} \bullet \underset{\approx}{\overline{\boldsymbol{E}}} \bullet \underset{\approx}{\boldsymbol{R}}^T \tag{4.33}$$

Both the Lagrange and Signorini strains are of Seth-Hill type with a Seth-Hill parameter $k = 2$. Recall that the general form for a Seth-Hill *spatial* strain uses the spatial (left) stretch:

$$\underset{\approx}{\varepsilon} = \frac{1}{k}[\underset{\approx}{\boldsymbol{V}}^k - \underset{\approx}{\boldsymbol{I}}] \tag{4.34}$$

Recalling that $\underset{\approx}{\boldsymbol{V}} = (\underset{\approx}{\boldsymbol{F}} \bullet \underset{\approx}{\boldsymbol{F}}^T)^{1/2}$, the Seth-Hill spatial strain can be written

$$\underset{\approx}{\varepsilon} = \frac{1}{k}[(\underset{\approx}{\boldsymbol{F}} \bullet \underset{\approx}{\boldsymbol{F}}^T)^{k/2} - \underset{\approx}{\boldsymbol{I}}] \tag{4.35}$$

Therefore, once again, choosing a Seth-Hill parameter that is an even integer has the advantage that no polar decomposition is required.

The **Eulerian** strain $\underset{\approx}{e}$ is a spatial strain measure corresponding to a Seth-Hill parameter $k = -2$:

$$\underset{\approx}{\varepsilon} = \frac{1}{2}[\underset{\approx}{\boldsymbol{I}} - (\underset{\approx}{\boldsymbol{F}} \bullet \underset{\approx}{\boldsymbol{F}}^T)^{-1}] = \frac{1}{2}[\underset{\approx}{\boldsymbol{I}} - (\underset{\approx}{\boldsymbol{F}}^{-T} \bullet \underset{\approx}{\boldsymbol{F}}^{-1})] \tag{4.36}$$

SPATIAL displacement gradients

The spatial displacement gradient tensor is just the derivative of the displacement with respect to the spatial (deformed) position vector:

$$\underset{\approx}{\boldsymbol{h}} = \frac{d\underset{\sim}{\boldsymbol{u}}}{d\underset{\sim}{\boldsymbol{x}}}, \qquad \text{or, in RCS, } h_{ij} = \frac{\partial u_i}{\partial x_j} \tag{4.37}$$

Recalling that $\underset{\sim}{\boldsymbol{u}} = \underset{\sim}{\boldsymbol{x}} - \underset{\sim}{\boldsymbol{X}}$. Differentiating this with respect to the spatial position vector gives

$$\underset{\approx}{\boldsymbol{h}} = \frac{d\underset{\sim}{\boldsymbol{u}}}{d\underset{\sim}{\boldsymbol{x}}} = \frac{d(\underset{\sim}{\boldsymbol{x}} - \underset{\sim}{\boldsymbol{X}})}{d\underset{\sim}{\boldsymbol{x}}} = \frac{d\underset{\sim}{\boldsymbol{x}}}{d\underset{\sim}{\boldsymbol{x}}} - \frac{d\underset{\sim}{\boldsymbol{X}}}{d\underset{\sim}{\boldsymbol{x}}} = \underset{\approx}{\boldsymbol{I}} - \underset{\approx}{\boldsymbol{F}}^{-1} \tag{4.38}$$

Solving for $\underset{\approx}{\boldsymbol{F}}^{-1}$ in Eq. (4.38) and substituting the result into Eq. (4.36) gives

$$\underset{\approx}{\varepsilon} = \frac{1}{2}[\underset{\approx}{\boldsymbol{h}} + \underset{\approx}{\boldsymbol{h}}^T - \underset{\approx}{\boldsymbol{h}}^T \bullet \underset{\approx}{\boldsymbol{h}}] \tag{4.39}$$

This is eq. (3.28) in the book.

Small displacement gradients

Recall from Eq. (4.1) that all Seth-Hill *linear* strain measures become equivalent if the linear stretch $\lambda = L/L_o$ is close to 1. Similarly, if the stretch tensor is close to the identity (i.e., if all of its eigenvalues are close to 1), then all reference strain measures are approximately equivalent, so we may define a small-stretch* reference strain tensor as

$$\underset{\approx}{\bar{\varepsilon}}^{\mathrm{SS}} = \underset{\approx}{\boldsymbol{U}} - \underset{\approx}{\boldsymbol{I}} \tag{4.40}$$

Likewise, we can define

$$\underset{\approx}{\varepsilon}^{\mathrm{SS}} = \underset{\approx}{\boldsymbol{V}} - \underset{\approx}{\boldsymbol{I}} \tag{4.41}$$

Here "SS" means small stretch (i.e., each principal stretch is approximately equal to 1). For small stretch deformations, any identity that was true only for a particular Seth-Hill parameter becomes true (approximately) for the small stretch tensor as well. For example, the volumetric strain is in general equal to the trace of the strain tensor only for logarithmic strain definitions. However, *for small stretches*, this statement holds approximately regardless of the strain definition.

Even if stretches are small, however, there might still be large rotations, so

$$\underset{\approx}{\bar{\varepsilon}}^{\mathrm{SS}} \neq \underset{\approx}{\varepsilon}^{\mathrm{SS}} \tag{4.42}$$

* The phrase "small stretch" does not mean the principal stretches are close to zero; it means they are close to 1.

In order to assert that *all* strain definitions are approximately equivalent, we must add the proviso that the rotation is small as well. This means that the deformation gradient tensor is approximately equal to the identity and therefore the displacement gradient tensor is approximately zero.

If the displacement gradients are small, then the last term in Eq. (4.25) is negligible, and Eq. (4.25) becomes

$$\underset{\approx}{\varepsilon}^{\mathrm{SDG}} = \frac{1}{2}(\underset{\approx}{\boldsymbol{H}} + \underset{\approx}{\boldsymbol{H}}^T) \tag{4.43}$$

Here, "SDG" stands for "small displacement gradients." For small displacement gradients, not only is the stretch small, but the rotation is small too. Therefore

$$\underset{\approx}{\varepsilon}^{\mathrm{SDG}} = \underset{\approx}{\boldsymbol{U}} - \underset{\approx}{\boldsymbol{I}}, \tag{4.44}$$

or

$$\underset{\approx}{\boldsymbol{U}} = \frac{1}{2}(\underset{\approx}{\boldsymbol{H}} + \underset{\approx}{\boldsymbol{H}}^T) + \underset{\approx}{\boldsymbol{I}} \tag{4.45}$$

For small displacement gradients, the infinitesimal rotation tensor is approximately

$$\underset{\approx}{\boldsymbol{R}}^{\mathrm{SDG}} = \frac{1}{2}(\underset{\approx}{\boldsymbol{H}} - \underset{\approx}{\boldsymbol{H}}^T) \tag{4.46}$$

This approximation must be handled carefully since this expression does not correspond to an orthogonal tensor.

Volume strain

In the section on the polar decomposition, we noted that the Jacobian J equals the determinant of stretch, which is simply the product of the principal stretches:

$$J = \lambda_1\lambda_2\lambda_3 \tag{4.47}$$

Furthermore, in the section on the deformation gradient, we showed that the ratio of deformed to initial volumes equals the Jacobian

$$J = \frac{V}{V_o} \tag{4.48}$$

Thus, the Jacobian is the volumetric analog of the linear stretch, $\lambda = L/L_o$. When a strain is defined using a particular choice k for the Seth-Hill strain parameter, then

$$\underset{\approx}{\bar{\varepsilon}} = \frac{1}{k}(\underset{\approx}{\boldsymbol{U}}^k - \underset{\approx}{\boldsymbol{I}}) \tag{4.49}$$

The *consistent* definition of volumetric strain uses the same value for the Seth-Hill parameter so that

$$\varepsilon_v = \frac{1}{k}(J^k - 1) \tag{4.50}$$

In this section, we will show how the consistent volumetric strain is related to invariants of the strain tensor — we will prove that it is *not* equal to the trace of the strain tensor except under conditions of small strain.

From Eq. (4.49), we know

$$\underset{\approx}{\boldsymbol{U}}^k = \underset{\approx}{\boldsymbol{I}} + k\bar{\underset{\approx}{\varepsilon}} \tag{4.51}$$

Therefore, taking the determinant,

$$J^k = \det(\underset{\approx}{\boldsymbol{I}} + k\bar{\underset{\approx}{\varepsilon}}) \tag{4.52}$$

Using a fundamental theorem from 3D matrix analysis, the determinant on the right hand side can be expanded to give

$$J^k = \det(\underset{\approx}{\boldsymbol{I}}) + \underset{\approx}{\boldsymbol{I}}^C : (k\bar{\underset{\approx}{\varepsilon}}) + \underset{\approx}{\boldsymbol{I}} : (k\bar{\underset{\approx}{\varepsilon}})^C + \det(k\bar{\underset{\approx}{\varepsilon}}) \tag{4.53}$$

where the superscript "C" denotes the cofactor. The cofactor of the identity is just the identity itself, so the second term on the right hand side simplifies to kI_1, where $I_1 \equiv \mathrm{tr}\bar{\underset{\approx}{\varepsilon}}$. Also, $(k\bar{\underset{\approx}{\varepsilon}})^C = k^2(\bar{\underset{\approx}{\varepsilon}})^C$, so the third term becomes $k^2 I_2$, where I_2 is the second invariant of strain, given by $\tilde{I}_2 = \mathrm{tr}(\bar{\underset{\approx}{\varepsilon}})^C$. Finally, the last term is just the third invariant I_3 times k^3. Thus, the above equation may be written

$$J^k = 1 + kI_1 + k^2 I_2 + k^3 I_3 \tag{4.54}$$

and the consistent volumetric strain becomes

$$\varepsilon_v = I_1 + kI_2 + k^2 I_3 \tag{4.55}$$

For arbitrary deformations, note that the volume strain will be identically equal to the trace of the strain only if $k=0$. In other words, the logarithmic (Hencky) strain measure is the *only* one whose consistent volume strain measure is given by the trace of the strain tensor. Of course, for small deformations, the strain measures become indistinguishable, so approximating the volumetric strain by the trace of the strain is acceptable in that case.

5. EXAMPLES: non-rotational deformations

Uniaxial Strain

For **uniaxial strain**, all particle motion occurs in one direction. Because there is no lateral material motion, the shape of the cross-sectional area is irrelevant — it might be a circular, square, or any other shape. For this reason, we may simplify the analysis without any loss in generality by presuming that the cross-section is rectangular and is therefore most easily described using a rectangular Cartesian system (RCS) for the coordinates.

Let's suppose that we set up the RCS basis so that the 1-direction is aligned with the direction of motion. The initial location of a material particle can be expressed in terms of our RCS as

$$\underset{\sim}{\boldsymbol{X}} = X_1\underset{\sim}{\boldsymbol{e}}_1 + X_2\underset{\sim}{\boldsymbol{e}}_2 + X_3\underset{\sim}{\boldsymbol{e}}_3 \tag{5.1}$$

After deformation, this particle deforms to a new location

$$\underset{\sim}{\boldsymbol{x}} = x_1\underset{\sim}{\boldsymbol{e}}_1 + x_2\underset{\sim}{\boldsymbol{e}}_2 + x_3\underset{\sim}{\boldsymbol{e}}_3 \tag{5.2}$$

where, for uniaxial strain,

$$x_1 = f(X_1) \tag{5.3}$$

$$x_2 = X_2 \tag{5.4}$$

$$x_3 = X_3 \tag{5.5}$$

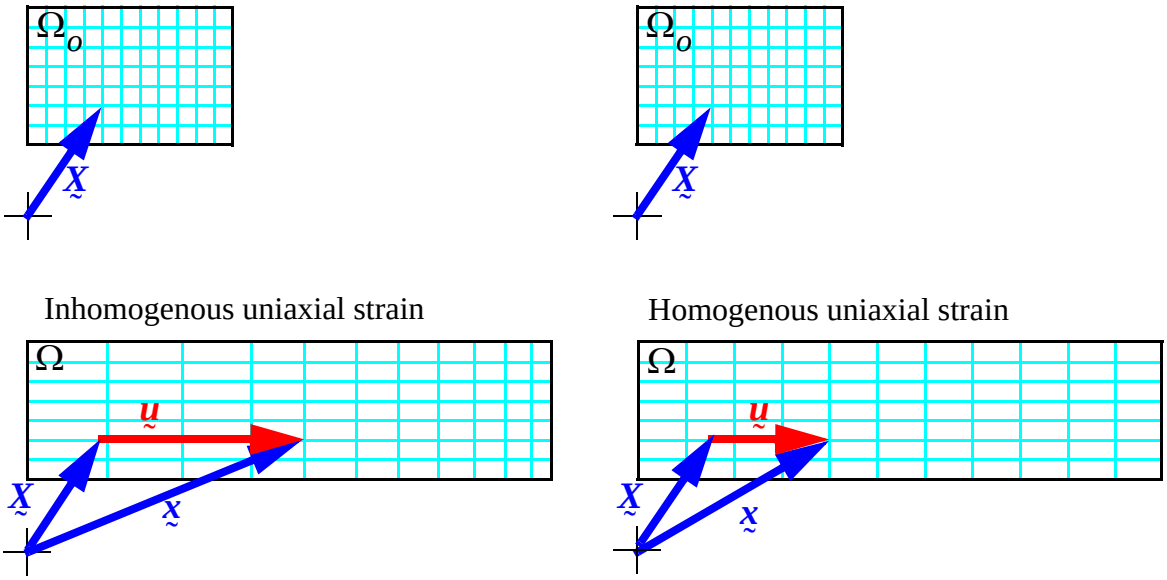
The last two of these equations state that no motion occurs transverse to the 1-direction. All motion is characterized by the 1D deformation mapping function f. Here, the function f depends on the nature of the loading. The deformation gradient is given by

$$[\underset{\sim}{\boldsymbol{F}}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } \lambda \equiv f'(x_1) \tag{5.6}$$

Note that the local "stretch" λ varies only along the axial direction. When the stretch is constant, the deformation is said to be *homogeneous* uniaxial strain. Suppose, for example, that a piston contains a gas. If the piston is slowly compressed from a length L_o to L, then the deformation will satisfy $x_1/X_1 = L/L_o$, or

$$x_1 = (L/L_o)X_1, \text{ and therefore } \lambda = L/L_o \tag{5.7}$$

If, on the other hand, the piston is *suddenly* compressed from a length L_o to L, then the material displacements at a given instant in time will be highly inhomogenous due to wave motion back and forth. Even if a deformation is equilibrated, it might not be homogenous. Consider, for example, a long cylinder of a compressible gel sitting on a table so that the axis of the cylinder points up. Then gravity will make material more compressed at the bottom of the cylinder than at the top.



The left hand side of the above figure shows *inhomogenous* uniaxial strain. Note that the material elements (i.e., the little squares of material) don't deform very much on the right side of the bar, but they are stretched by more than 400% on the left hand side. You might expect a deformation somewhat like this if you hang a long cylinder of cork* from the ceiling. For *homogenous* uniaxial strain, each little material element stretches by the same amount everywhere.

Uniform spherical contraction/expansion

Consider a deformation for which

$$\underset{\sim}{\boldsymbol{x}} = \lambda \underset{\sim}{\boldsymbol{X}} \tag{5.8}$$

In other words, each initial material vector changes length by the same proportion. Then

$$\underset{\approx}{\boldsymbol{F}} = \lambda \underset{\approx}{\boldsymbol{I}} \tag{5.9}$$

* Cork is a material whose Poisson's ratio is approximately zero. Consequently, it would have no off-axis (transverse) contractions. This constraint is needed to ensure uniaxial *strain* conditions.

Non-uniform spherical deformation

Consider a deformation in which spherical coordinates $\{R, \Theta, \Phi\}$ are used to locate the initial position of a point and spherical coordinates r, θ, ϕ identify the deformed location of that point. A spherical deformation is one for which

$$r = f(R), \qquad \theta = \Theta, \qquad \text{and} \qquad \phi = \Phi \tag{5.10}$$

Triaxial strain

Triaxial strain is a homogenous axially symmetric deformation. It differs from uniaxial strain only in that lateral deformations are permitted. Specifically, the deformation gradient is of the form

$$\underset{\approx}{\boldsymbol{F}} = \lambda_A \underset{\approx}{\boldsymbol{P}}_A + \lambda_T \underset{\approx}{\boldsymbol{P}}_T \tag{5.11}$$

Here, the subscripts "A" and "T" stand for "axial" and "transverse" respectively, and the coefficient tensors are projection operations defined

$$\underset{\approx}{\boldsymbol{P}}_A \equiv \underset{\sim}{\boldsymbol{n}}\underset{\sim}{\boldsymbol{n}} \tag{5.12}$$

and

$$\underset{\approx}{\boldsymbol{P}}_T \equiv \underset{\approx}{\boldsymbol{I}} - \underset{\sim}{\boldsymbol{n}}\underset{\sim}{\boldsymbol{n}} \tag{5.13}$$

The vector $\underset{\sim}{\boldsymbol{n}}$ is a unit vector in the direction of the symmetry axis, and $\underset{\sim}{\boldsymbol{n}}\underset{\sim}{\boldsymbol{n}}$ denotes the dyadic multiplication of $\underset{\sim}{\boldsymbol{n}}$ with itself (the result is a second-order tensor with ij components given by $n_i n_j$. When $\underset{\approx}{\boldsymbol{P}}_A$ acts on an arbitrary vector $\underset{\sim}{\boldsymbol{v}}$, the result is the part of that vector in the direction of the symmetry axis. When $\underset{\approx}{\boldsymbol{P}}_T$ acts on the vector, the result is the part transverse (perpendicular) to the symmetry axis. Stated mathematically, for any vector, $\underset{\sim}{\boldsymbol{v}}$

$$\underset{\approx}{\boldsymbol{P}}_A \bullet \underset{\sim}{\boldsymbol{v}} = (\underset{\sim}{\boldsymbol{v}} \bullet \underset{\sim}{\boldsymbol{n}})\underset{\sim}{\boldsymbol{n}} \tag{5.14}$$

$$\underset{\approx}{\boldsymbol{P}}_A \bullet \underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{v}} - (\underset{\sim}{\boldsymbol{v}} \bullet \underset{\sim}{\boldsymbol{n}})\underset{\sim}{\boldsymbol{n}} \tag{5.15}$$

We have stated that triaxial strain is a being here described as a *homogenous* deformation, which means that the deformation gradient tensor $\underset{\approx}{\boldsymbol{F}}$ is spatially constant and the definition of the deformation gradient, $d\underset{\sim}{\boldsymbol{x}} = \underset{\approx}{\boldsymbol{F}} \bullet d\underset{\sim}{\boldsymbol{X}}$ may therefore be integrated to give

$$\underset{\sim}{\boldsymbol{x}} = \underset{\approx}{\boldsymbol{F}} \bullet \underset{\sim}{\boldsymbol{X}} + \underset{\sim}{\boldsymbol{C}}, \tag{5.16}$$

where $\underset{\sim}{\boldsymbol{C}}$ is an integration constant representing rigid translation. In light of Eqs. (5.11), (5.14) and (5.15), the deformation mapping function can be written

$$\underset{\sim}{\boldsymbol{x}} = \lambda_A(\underset{\sim}{\boldsymbol{X}} \bullet \underset{\sim}{\boldsymbol{n}})\underset{\sim}{\boldsymbol{n}} + \lambda_T[\underset{\sim}{\boldsymbol{X}} - (\underset{\sim}{\boldsymbol{X}} \bullet \underset{\sim}{\boldsymbol{n}})\underset{\sim}{\boldsymbol{n}}] + \underset{\sim}{\boldsymbol{C}} \tag{5.17}$$

In terms of RCS coordinates with the 1-direction aligned with $\underset{\sim}{\boldsymbol{n}}$, the deformation gradient and the mapping function may be written

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} \lambda_A & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & \lambda_T \end{bmatrix} \tag{5.18}$$

and

$$x_1 = \lambda_A X_1 + C_1 \tag{5.19}$$

$$x_2 = \lambda_T X_2 + C_2 \tag{5.20}$$

$$x_3 = \lambda_T X_3 + C_3 \tag{5.21}$$

Here, X_T denotes the distance of $\underset{\sim}{\boldsymbol{X}}$ from the symmetry axis.

Note that uniform spherical deformation (discussed on page 39) is a special case of triaxial deformation in which $\lambda_A = \lambda_L = \lambda$ and $\underset{\sim}{\boldsymbol{C}} = \underset{\sim}{\boldsymbol{0}}$. Also, homogenous uniaxial strain is a special case of triaxial deformation in which $\lambda_A = \lambda$ and $\lambda_T = 1$.

6. Axisymmetric deformations

Constricted rod

Consider a deformation in which a rod expands or contracts to a an axisymmetric rod of varying profile. Let $\underset{\sim}{\boldsymbol{a}}$ denote a unit vector that defines the axis of symmetry. Any vector $\underset{\sim}{\boldsymbol{v}}$ can always be decomposed into parts parallel and perpendicular to the axis as follows

$$\underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{v}}^{\text{axial}} + \underset{\sim}{\boldsymbol{v}}^{\text{lateral}},$$
$$\text{where } \underset{\sim}{\boldsymbol{v}}^{\text{axial}} = \underset{\sim}{\boldsymbol{a}}(\underset{\sim}{\boldsymbol{a}} \bullet \underset{\sim}{\boldsymbol{v}}) \text{ and } \underset{\sim}{\boldsymbol{v}}^{\text{lateral}} = \underset{\sim}{\boldsymbol{v}} - \underset{\sim}{\boldsymbol{v}}^{\text{axial}} \tag{6.1}$$

This projection decomposition can be applied to both the initial and deformed position vectors. Under an axisymmetric deformation, the axial part of the deformed vector is proportional to the axial part of the undeformed vector and the lateral part of the deformed vector is proportional to the lateral part of the initial vector. The two proportionality constants are not required to be equal to each other. Furthermore, the proportionality constants are permitted to be functions of the axial position. Thus, for the axial part, we have

$$\underset{\sim}{\boldsymbol{x}}^{\text{axial}} = f(X_a)\underset{\sim}{\boldsymbol{X}}^{\text{axial}} \tag{6.2}$$

where

$$X_a = \underset{\sim}{\boldsymbol{X}} \bullet \underset{\sim}{\boldsymbol{a}} \tag{6.3}$$

Both $\underset{\sim}{\boldsymbol{x}}^{\text{axial}}$ and $\underset{\sim}{\boldsymbol{X}}^{\text{axial}}$ are proportional to $\underset{\sim}{\boldsymbol{a}}$. Consequently,

$$x_a \equiv \underset{\sim}{\boldsymbol{a}} \bullet \underset{\sim}{\boldsymbol{x}} = f(X_a)X_a \tag{6.4}$$

Thus, rather than using the function f, we can say

$$x_a = g(X_a) \tag{6.5}$$

For the lateral part, we have

$$\underset{\sim}{\boldsymbol{x}}^{\text{lateral}} = h(X_a)\underset{\sim}{\boldsymbol{X}}^{\text{lateral}} \tag{6.6}$$

Putting it all together, the mapping function is

$$\underset{\sim}{\boldsymbol{x}} = \underset{\sim}{\boldsymbol{a}}g(X_a) + (\underset{\sim}{\boldsymbol{X}} - \underset{\sim}{\boldsymbol{a}}X_a)h(X_a), \tag{6.7}$$

In terms of a coordinate system for which $\underset{\sim}{\boldsymbol{E}}_1 = \underset{\sim}{\boldsymbol{a}}$, this becomes

$$x_1 = g(X_1) \tag{6.8}$$

$$x_2 = h(X_1)X_2 \tag{6.9}$$

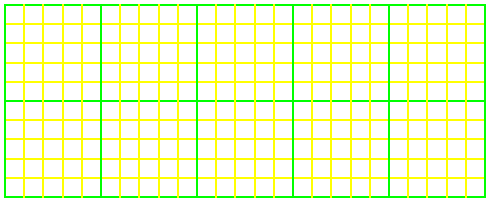
$$x_3 = h(X_1)X_3 \tag{6.10}$$

Differentiating the mapping function, Eq. (6.7), with respect to $\underset{\sim}{\boldsymbol{X}}$ gives the deformation gradient tensor

$$\underset{\approx}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{a}}g'(X_a) + (\underset{\sim}{\boldsymbol{X}} - \underset{\sim}{\boldsymbol{a}}X_a)\underset{\sim}{\boldsymbol{a}}h'(X_a) + (\underset{\approx}{\boldsymbol{I}} - \underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{a}})h(X_a) \tag{6.11}$$

which, in terms of an aligned basis becomes

Initial configuration



Deformed configuration

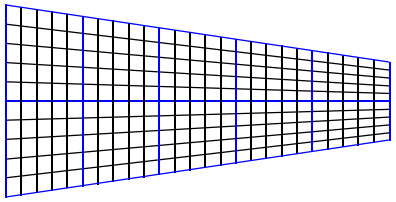


Figure 6.1. Axisymmetric deformation. In this figure, $g(X_1) = \frac{4}{5}X_1$ and $h(X_1) = 1 - \frac{3}{5}\left(\frac{X_1}{25}\right)$

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} g'(X_1) & 0 & 0 \\ X_2[h'(X_1)] & h(X_1) & 0 \\ X_3[h'(X_1)] & 0 & h(X_1) \end{bmatrix}, \text{ where } f = f(X_1) \tag{6.12}$$

Note that along the centerline the deformation gradient is given by

$$\underset{\approx}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{a}} g'(X_a) + (\underset{\approx}{\boldsymbol{I}} - \underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{a}})h(X_a) \tag{6.13}$$

or

$$\underset{\approx}{\boldsymbol{F}} = \Lambda(\underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{a}}) + \lambda(\underset{\approx}{\boldsymbol{I}} - \underset{\sim}{\boldsymbol{a}}\underset{\sim}{\boldsymbol{a}}), \text{ where } \Lambda = g'(X_a) \text{ and } \lambda = h(X_a) \tag{6.14}$$

In terms of an aligned basis, the deformation gradient along the centerline is given by

$$[\underset{\approx}{\boldsymbol{F}}] = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tag{6.15}$$

The Jacobian is given by

$$J = \Lambda\lambda^2 \tag{6.16}$$

7. EXAMPLES: large rotation problems

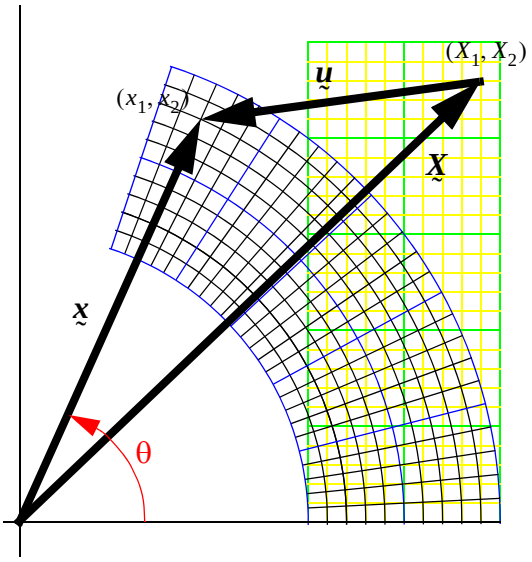
Bar bending

Note: this is the same as the example on page 96 of the textbook.

A bar, initially of height H, bends into curved wedge segment as shown until the top surface of the bar is oriented at angle θ_{max}. The geometry of this deformation can be alternatively described by specifying the radius R of the centerline and the stretch λ^c of the center line (i.e., its deformed length divided by its initial length). One can easily switch back and forth between these two specification methods by using the relation

$$R\theta_{max} = \lambda^c H, \tag{7.1}$$

which follows from the definition of stretch.



For this deformation, we seek:

1. The mapping function.
2. The deformation gradient.
3. The polar rotation tensor $\underset{\approx}{\boldsymbol{R}}$ and reference (right) stretch $\underset{\approx}{\boldsymbol{U}}$.
4. The Lagrange strain

Let $\underset{\sim}{\boldsymbol{X}} = X_1\underset{\sim}{\boldsymbol{E}}_1 + X_2\underset{\sim}{\boldsymbol{E}}_2$ denote the initial position of a point in the body, and let $\underset{\sim}{\boldsymbol{x}} = x_1\underset{\sim}{\boldsymbol{E}}_1 + x_2\underset{\sim}{\boldsymbol{E}}_2$ denote the deformed location of that same point.

First let's find the mapping function. In other words, we seek formulas for the deformed coordinates (x_1, x_2) of a point expressed in terms of the *initial* coordinates (X_1, X_2) of that same point. Any initially vertical line can be identified by its initial abscissa value, X_1. Each initially vertical line bends into the shape of an arc of radius X_1, with an arc length given by θX_1, where θ is the arc angle. The arc angle increases in proportion to X_2. Therefore, a constant α exists such that

$$\theta = \alpha X_2 \tag{7.2}$$

The arc length of a point originally located at (X_1, X_2) must be given by

$$s = \alpha X_1 X_2 \tag{7.3}$$

Applying this equation to the center line gives

$$R\theta_{\max} = \alpha RH \tag{7.4}$$

Therefore

$$\alpha = \frac{\theta_{\max}}{H} = \frac{\lambda^c}{R} \tag{7.5}$$

and Eq. (7.2) and Eq. (7.3) becomes

$$\theta = \frac{X_2}{H}\theta_{\max} \tag{7.6}$$

For example, if you are located 2/3 of the way up the bar in the undeformed configuration, then you will be 2/3 of the way along the arc in the deformed configuration.

Consider a point (X_1, X_2), not necessarily on the center line. After deformation, this point deforms to a new point (x_1, x_2) located a distance $r = X_1$ from the origin. Consequently,

$$x_1 = r\cos\theta = X_1\cos\left(\frac{X_2}{H}\theta_{\max}\right) \tag{7.7}$$

$$x_2 = r\sin\theta = X_1\sin\left(\frac{X_2}{H}\theta_{\max}\right) \tag{7.8}$$

The deformation gradient is given by

$$F_{11} = \frac{\partial x_1}{\partial X_1} = \cos\left(\frac{X_2}{H}\theta_{\max}\right) \qquad F_{12} = \frac{\partial x_1}{\partial X_2} = -\lambda^c\frac{X_1}{R}\sin\left(\frac{X_2}{H}\theta_{\max}\right)$$
$$F_{21} = \frac{\partial x_2}{\partial X_1} = \sin\left(\frac{X_2}{H}\theta_{\max}\right) \qquad F_{22} = \frac{\partial x_2}{\partial X_2} = \lambda^c\frac{X_1}{R}\cos\left(\frac{X_2}{H}\theta_{\max}\right) \tag{7.9}$$

Assembled into matrix form,

$$[F] = \begin{bmatrix} c & -\Gamma s \\ s & \Gamma c \end{bmatrix}, \tag{7.10}$$

To save "ink" in upcoming calculations, we have defined

$$\Gamma \equiv \frac{X_1 \theta_{\max}}{H} \equiv \lambda^c \frac{X_1}{R}, \tag{7.11}$$

$$c \equiv \cos\left(\frac{X_2}{H}\theta_{\max}\right) \equiv \cos\left(X_2 \frac{\lambda^c}{R}\right) \tag{7.12}$$

$$s \equiv \sin\left(\frac{X_2}{H}\theta_{\max}\right) = \sin\left(X_2 \frac{\lambda^c}{R}\right) \tag{7.13}$$

The Jacobian equals to the ratio of the deformed to undeformed volumes and can be computed by

$$J = \det \underset{\approx}{F} = \Gamma \tag{7.14}$$

The polar decomposition gives

$$[R] = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad \text{and} \qquad [U] = \begin{bmatrix} 1 & 0 \\ 0 & \Gamma \end{bmatrix} \tag{7.15}$$

Note that this deformation consists of a stretch Γ in the 2-direction, followed by a rotation into the final deformed orientation.

The Lagrange strain is

$$[E] = \frac{1}{2}\{[F]^T[F] - [I]\} = \frac{1}{2}\{[U]^2 - [I]\} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}(\Gamma^2 - 1) \end{bmatrix} \tag{7.16}$$

The reference logarithmic strain is

$$[\bar{\varepsilon}] = \ln[U] = \begin{bmatrix} 0 & 0 \\ 0 & \ln\Gamma \end{bmatrix} \tag{7.17}$$

The logarithmic volume strain is given by

$$\varepsilon_v = \mathrm{tr}\,\underset{\approx}{\bar{\varepsilon}} = \ln\Gamma \tag{7.18}$$

The displacement gradient is given by

$$[H] = [F] - [I] = \begin{bmatrix} (c-1) & -\Gamma s \\ s & (\Gamma c - 1) \end{bmatrix} \tag{7.19}$$

For this problem, the wrong-headed rotation measure becomes

$$[\Omega] = \begin{bmatrix} (c-1) & -\Gamma s \\ s & \Gamma\left(c - \frac{\Gamma}{2}\right) \end{bmatrix} \tag{7.20}$$

Unlike the polar rotation, there is not really any physical interpretation for this tensor. For pure stretch deformations, it equals the difference between the engineering strain (Seth-Hill parameter $k=1$) and the Lagrange strain (Seth-Hill parameter $k=2$). Under sufficiently large stretches, this difference will be neither zero nor the identity even in the absence of rotation.

Homogenous shear

<need to add this section>

Torsion

<need to add this section>

Vortex

<need to add this section>