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A model for statistical variation of fracture properties in a continuum mechanics code

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ABSTRACT

Continuum mechanics codes modeling failure of materials historically have considered those materials to be homogeneous, with all elements of a material in the computation having the same failure properties. This is, of course, unrealistic but expedient. But as computer hardware and software has evolved, the time has come to investigate a higher level of complexity in the modeling of failure. The Johnson–Cook fracture model is widely used in such codes, so it was chosen as the basis for the current work. The CTH finite difference code is widely used to model ballistic impact and penetration, so it also was chosen for the current work.

The model proposed here does not consider individual flaws in a material, but rather varies a material's Johnson–Cook parameters from element to element to achieve inhomogeneity. A Weibull distribution of these parameters is imposed, in such a way as to include a size effect factor in the distribution function. The well-known size effect on the failure of materials must be physically represented in any statistical failure model not only for the representations of bodies in the simulation (e.g., an armor plate), but also for the computational elements, to mitigate element resolution sensitivity of the computations.

The statistical failure model was tested in simulations of a Behind Armor Debris (BAD) experiment, and found to do a much better job at predicting the size distribution of fragments than the conventional (homogeneous) failure model. The approach used here to include a size effect in the model proved to be insufficient, and including correlated statistics and/or flaw interactions may improve the model.

1. Introduction

Continuum mechanics codes using Lagrangian frames and those using Eulerian frames have achieved great success in modeling ballistic impact and penetration. One widely-used model for determining material failure in such simulations is the Johnson–Cook fracture model [1], which computes a path-dependent failure using the following relation for current failure strain:

$$\varepsilon^f = \left[D_1 + D_2 e^{D_3\sigma^*}\right][1 + D_4 \ln\dot{\varepsilon}^*][1 + D_5 T^*]. \quad (1)$$

In Eq. (1), D_1, D_2, D_3, D_4 and D_5 are material constants. σ^* is the ratio of mean stress to the von Mises equivalent stress, $\dot{\varepsilon}^*$ is the non-dimensional strain-rate, and T^* is the homologous temperature.

Historically, these continuum mechanics codes applied the Johnson–Cook fracture model in a deterministic fashion throughout the problem domain; i.e., for a particular material, the same set of material constants applies to every element or cell[1] containing that material. However, because of heterogeneity, real materials are not perfectly deterministic, but instead exhibit variations of properties, for example fracture properties, throughout the volume of material. A recent approach [2,3] applied statistical variations of fracture properties to materials in a Lagrangian frame. Here, the approach is applied to the Johnson–Cook fracture model in an Eulerian code, CTH [4], and extended to account for size effects, a step toward the ultimate goal of an element-size-invariant failure criterion for consistent predictions across a spectrum of system geometries. The goal of the present work is to illustrate the dramatic improvement in the character of results (especially irregular failure patterns) achieved when heterogeneous failure properties and scale effects are incorporated into a simulation. This should lead to the realization

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[1] The term element is usually applied in the context of a Lagrangian formulation, while the term cell usually applies to the Eulerian computational unit. In the following discussions, the term element will be used for both applications, for simplicity.

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that better experimental characterization of the failure distribution is needed, and provide motivation to obtain that data.

2. Statistical variation of the initial failure strain

Identification of constraints is important if one seeks to statistically perturb material parameters. If, for example, a Weibull perturbation is applied to a material parameter, then realizations of that parameter will vary from zero to infinity, and such variation must not violate any fundamental constraints on material parameters.

Consider a specimen at room temperature in a state of zero pressure (mean stress), being strained at a rate of 1/s. The strain at failure of such a specimen will be termed initial failure strain, ε_0^f. Under these conditions, Eq. (1) reduces to the following expression for the initial failure strain:

$$\varepsilon_0^f = D_1 + D_2. \tag{2}$$

Admissible Johnson–Cook material parameters must obey the constraint D1 + D2 > 0 to ensure that a positive failure strain is required to induce failure. Satisfaction of this constraint has been confirmed for numerous materials for which Johnson–Cook data sets are available [5]. However, some of those materials have negative values for D1. Therefore, D1 and D_2 may not be individually perturbed according to a Weibull distribution since there will exist realizations violating the constraint $D_1 + D_2 > 0$. However, their sum, the initial failure strain ε_0^f, can be perturbed via a Weibull distribution (or any other distribution for which negative realizations are impossible[2]).

Weibull [6] considered the statistics of failure events (i.e., failure of systems), for example the fatigue life of a rotating steel beam.[3] Here, the applicability of Weibull statistics at an *element level* within a shock physics code is explored under the following progression of assumptions:

The failure strain ε^f in an elemental volume of the system is strongly dependent on the criticality[4] of the flaws in that element.

The criticality of an isolated flaw is not necessarily Weibull-distributed, but a Weibull distribution is reasonable for an ensemble of flaws of random orientations if the population contains many small flaws and relatively few large flaws [7].

The failure strains of the ensemble of elements making up the system are also Weibull-distributed [7].

Since $\varepsilon^f = f(\varepsilon_0^f)$, the initial failure strain in each of the many volume elements making up the system is also Weibull-distributed. This assumption differs from the approach in [3], as discussed later.

Therefore the initial failure strain will be Weibull-distributed throughout the elements of a material.[5]

Eq. (1) does not explicitly contain the initial failure strain; to use a statistically varying initial failure strain, Eq. (1) is rearranged algebraically to the following form:

$$\varepsilon^f = \left[\varepsilon_0^f - D_2\left(1 - e^{D_3\sigma^*}\right)\right][1 + D_4 \ln\dot{\varepsilon}^*][1 + D_5 T^*]. \tag{3}$$

As mentioned, the failure strain in the standard Johnson–Cook model is deterministic. However, variations in micromorphology of a material lead to variations in failure strain. Probabilities associated with such variations lead to a dependence of failure strain on the specimen size. Larger samples are more likely to contain a critically oriented or critically large flaw, making larger samples statistically more prone to failure at a given strain. Below, a mathematical framework [7] is developed to account for these macroscale effects of variability in flaw morphology without requiring actual details about crack sizes, shapes, orientations, or clustering.

Consider a sample of volume V containing exactly one flaw. Let the sample be subjected to a prescribed strain, ε. Whether or not the sample will fail is uncertain because of uncertainties in flaw morphology such as crack orientation, size, or shape. If, for example, the strain state is tensile in one direction and compressive in another, the sample is certainly safe from failure if the crack normal is aligned with the compressive direction, but flaw orientations are unknown. Even when all principal strains are compressive, a flaw can fail under shear if it is critically oriented and sufficiently large. However, flaw size is unknown.

Regardless of the basis of uncertainty, let $g(\varepsilon)$ symbolically denote the probability that the sample is safe from failure at the applied strain ε (this single-flaw probability is not expected to be Weibull-distributed). Under a non-interaction assumption, a sample containing N flaws is safe from failure only if every flaw in the sample is safe from failure, giving the probability that the sample is safe to be $P_s = [g(\varepsilon)]^N$. This is expected to be an upper-bound since flaw interactions are expected to reduce the likelihood that the sample is safe. For elastic properties, a non-interaction assumption is valid to very high crack densities [8], but such cannot be assumed for failure properties.

Let $n = N/V$ denote the flaw density and let P_f denote the probability of failure. Thus, for the non-interaction model,

$$P_f = 1 - P_s = 1 - [g(\varepsilon)]^N = 1 - [g(\varepsilon)]^{nV}. \tag{4}$$

To allow for the effect of flaw interactions in an approximate way, suppose that an increase in flaw density causes P_f to increase in a way similar to intensifying the strain in a non-interaction model. Then the non-interaction model can be generalized to account for flaw interactions by multiplying the strain by an intensifier function $H(n)$ to give

$$P_f = 1 - \{g[\varepsilon \cdot H(n)]\}^{nV}. \tag{5}$$

The strain intensifier function $H(n)$ is expected to be a monotonically increasing function of crack density so that an increase in crack density would lead to an increase in apparent strain in a non-interaction model and, therefore, an increase in P_f.

Unfortunately, both the g and H functions, as well as the flaw density n, are unknowable from a practical perspective. As previously argued, a Weibull distribution is to be assumed for the multiple-flaw elements. The Weibull distribution function [6] is

$$P_f = 1 - \exp[-\phi(\varepsilon)], \tag{6}$$

where $\phi(\varepsilon)$ is a material function to be measured in the laboratory by repeated testing of the strain at failure. Applying the definition of a Weibull distribution, the material function in Eq. (6) would be of the form

$$\phi(\varepsilon) = \left(\frac{\varepsilon}{a}\right)^m. \tag{7}$$

[2] For this reason, a Gumbel distribution would be inappropriate since its realizations have no lower bound.

[3] Subsequently, system will refer to a component of a simulation, e.g., an armor plate or a penetrator.

[4] Criticality is the propensity (or probability) of a flaw to initiate failure of the system under a given state of stress, for example the size of the flaw might affect its criticality.

[5] To be sure, these assumptions are tenuous, as little or no data exists on which to base an understanding of the correct distribution of flaws in a material. And the distribution may well be different for different materials, e.g., ceramic versus metal. Nontheless, applying a Weibull distribution to the initial failure strain is a reasonable starting point, from which the efficacy of such a statistical modeling approach may be judged. A desirable outcome of the present work is to motivate experimenters to obtain such data.

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The constant (positive) divisor a is called the scale parameter, and the (positive) exponent m is termed the shape factor. The genesis of these terms will become apparent in the next section. In the engineering literature, and in the remainder of discussions here, the shape factor is called the Weibull modulus.

The dependence of failure probability P_f on the number of flaws in a brittle material, and hence on sample size V, is clear. However, this is a tenuous basis for defining the failure probability of a ductile metal. Nontheless, it is a reasonable starting point for testing the effectiveness of including heterogeneity in a material in an Eulerian simulation. From Eqs. (5) and (6), the failure probability is taken to be of the form:

$$P_f = 1 - \{\exp[-\phi(\varepsilon)]\}^{nV}. \tag{8}$$

The scale parameter a can be eliminated as an explicit parameter by noting that the median failure strain $\overline{\varepsilon}$ for a sample of size $\overline{V}$ corresponds to $P_f = ½$. Setting $\varepsilon = \overline{\varepsilon}$, $V = \overline{V}$, and $P_f = ½$ in Eqs. (7) and (8) gives the following relationship between the scale parameter and the median failure strain:

$$a = \overline{\varepsilon}\left[\frac{\ln 2}{n\overline{V}}\right]^{-1/m}. \tag{9}$$

After rewriting Eq. (8) in terms of the probability of survival, P_s, where $P_s = 1 - P_f$, Eqs. (7) and (9) are substituted, and the survival probability becomes

$$P_s = 2^{-(V/\overline{V})}(\varepsilon/\overline{\varepsilon})^m. \tag{10}$$

A random realization of failure strain may be found by replacing P_s in Eq. (10) by a random number R uniform on the interval $(0 \le R \le 1)$, and solving for ε to obtain

$$\varepsilon = \overline{\varepsilon}\left[\frac{\ln R}{\ln 1/2}\right]^{\frac{1}{m}}\left[\frac{V}{\overline{V}}\right]^{-\frac{1}{m}}. \tag{11}$$

This equation may be applied to perturb the value of any parameter in a material model that is confirmed in the laboratory to vary according to a Weibull distribution. For scoping purposes while awaiting further data on the distributions of failure strain, ε is interpreted to be the initial failure strain parameter ε_0^f in Eq. (3). Accordingly, the Johnson–Cook fracture model is employed using a statistically varying value of the initial failure strain parameter generated by

$$\varepsilon_0^f = \overline{\varepsilon}_0^f\left[\frac{\ln R}{\ln 1/2}\right]^{\frac{1}{m}}\left[\frac{V}{\overline{V}}\right]^{-\frac{1}{m}}. \tag{12}$$

Recalling Eq. (2), the median value of the distribution of initial failure strains ($\overline{\varepsilon}_0^f$) is assumed equal to the library value ($D_1 + D_2$), from [5] for example.[6] Therefore, the volume $\overline{V}$ is taken as the volume of the homogeneously loaded gage section[7] of the test sample used to determine the constants D_1 and D_2. Each element in a calculation is assigned a distinct failure strain according to Eq. (12). Therefore, the volume V is the element volume.

Incidentally, the perturbation of the zero-pressure shear failure strain ε_0^f leads to a degree of uncertainty in shear failure strain that is the same at all pressures. The perturbation essentially introduces uncertainty in ordinates of the shear failure envelope, as illustrated in Fig. 1a. Since Weibull realizations range from zero to infinity, unrealistically large shear failure strain realizations are theoretically possible with the present approach. In parallel with the current efforts, Brannon et al. [3] applied Eq. (11) in a fundamentally different way. They applied a Weibull distribution to the zero-shear hydrostatic tensile strength, rather than applying the Weibull perturbation to the zero-pressure shear strength as is done here. Their approach leads to a lessening of uncertainty with pressure

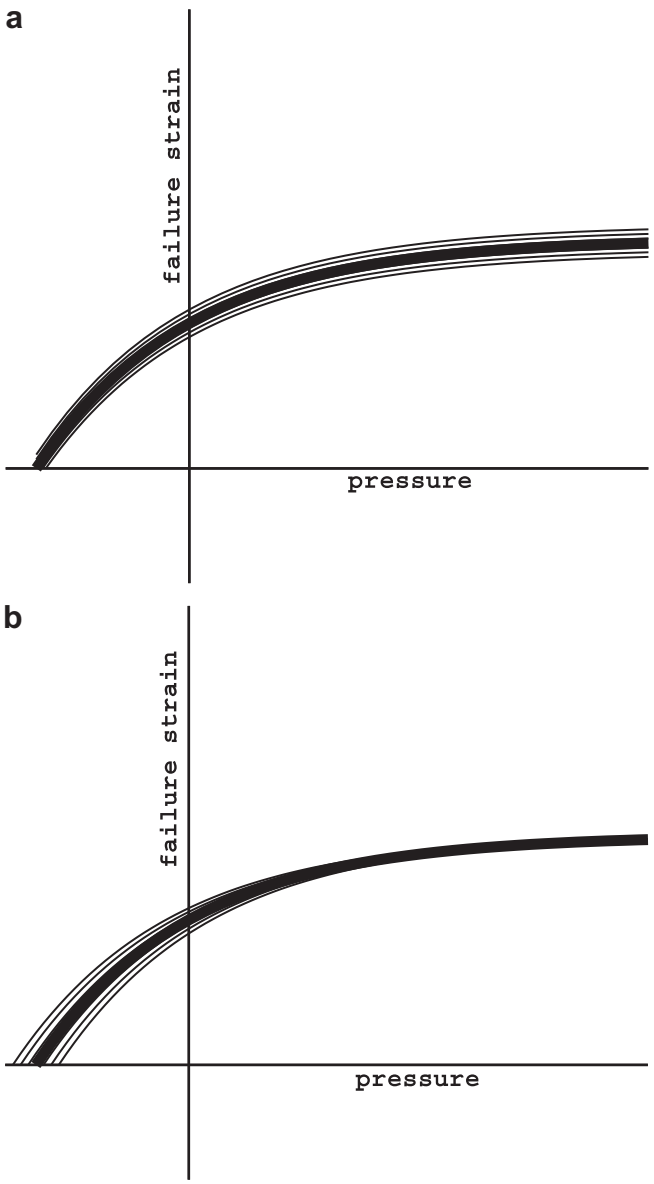

Fig. 1. Realizations of strength curves obtained by perturbing *(a)* the ordinate and *(b)* the abscissa of the reference curve (thick line). The current study explores ordinate perturbations of type "a". A concurrent study (3) has focused on abscissa perturbations of type "b", which corresponds to uncertainty decreasing with pressure and which prevents unrealistically large realizations of shear strength.

[6] Lacking experimental data on the actual distribution of initial failure strains in a material, an assumption is made that the median initial failure strain is equal to the library value of $(D_1 + D_2)$. That implies an assumption that many tests were conducted to evaluate the parameters D_1 and D_2, and that the median values of those tests were reported and entered into the CTH library.

[7] The present analysis presumes nominally homogeneous loading because elements in a discretized simulation are individually treated as homogeneously loaded. However, identifying an appropriate reference volume is difficult because laboratory experiments rarely involve homogeneous loading. Hence, the gage volume is only an approximation that can be improved by simulating the actual experiment. Our conjecture is that volume-based size effects at the element level could easily lead to different size effects at the larger scale of the experiment. Moreover, recent work (not reported here) suggests that the element-level size effect may require revision to approximate flaw interactions (later, size effect will be discussed in more detail).

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(Fig. 1b) and prevents infinite shear strengths. Exploring the relative merits of these different approaches requires further experimental data for the strength distributions at a variety of stress triaxialities.

Dørum et al. [9] employed another approach, limiting plastic work (which would become a limiter on plastic strain here). Such a limiter could mitigate the scale issues discussed later. Although the work of Dørum et al involved quasi-static loading, their approach (of limiting plastic work) could be a very practical way to introduce rate-dependent ductility. Since stress increases in high-rate loading, a work-based failure theory to would yield a lower plastic strain to failure at high rates.

As will be soon demonstrated, perturbing failure properties leads to more realistic localized failure. Moreover, this approach can help mitigate the mesh sensitivity that is well known to corrupt the solutions of classical deterministic damage models. Elements on a refined mesh are smaller and therefore stronger, on average, than elements on a coarse mesh. However, since there are more elements on a refined mesh, the probability of the onset of failure somewhere in a finite domain is the same as for a coarse mesh covering the same domain.

3. Sample distributions

Eqs. (6) and (7) constitute the Weibull cumulative distribution function (CDF), the probability of failure of the element at, or prior to, a strain of ε:

$$F(\varepsilon) = 1 - \exp\left[-\frac{\varepsilon}{a}\right]^m. \tag{13}$$

The derivative of the CDF gives the Weibull probability density function (PDF),

$$f(\varepsilon) = \frac{m}{a}\left(\frac{\varepsilon}{a}\right)^{m-1}\exp\left[-\frac{\varepsilon}{a}\right]^m. \tag{14}$$

The integral of the PDF gives the probability that the failure strain of the element lies between the limits of integration. The hazard function[8] gives the *instantaneous failure rate* (failures per unit strain increment) at a strain of ε. The hazard function is related to the CDF and PDF through the following expression:

$$h(\varepsilon) = \frac{f(\varepsilon)}{1 - F(\varepsilon)}. \tag{15}$$

Hence the Weibull hazard function is

$$h(\varepsilon) = \frac{m}{a}\left(\frac{\varepsilon}{a}\right)^{m-1}. \tag{16}$$

One can easily see from Eq. (16) that for a Weibull modulus $m > 1$, the hazard function increases with increasing strain (ε). That is to say, as the strain increases, an increasing number of elements will fail during each strain increment. For $m = 1$, the Weibull function reduces to the exponential distribution and the hazard function is constant, meaning failures occur at a constant rate no matter how high the strain (this is the "memoryless" property of an exponential distribution). For a Weibull modulus $m < 1$, the hazard function decreases with increasing strain. In this case, elements get more resistant to failure as strain increases. Failure rate normally increases with increasing strain so, for most applications involving failure strain, $m > 1$ seems appropriate.

Fig. 2 shows sample distributions of initial failure strain as they would be produced in a CTH problem generation, for various values of the Weibull modulus. Examples of distributions for $m < 1$ and $m = 1$ are included for completeness. The failure strains shown are actually multiples of the median failure strain $\bar{\varepsilon}_0^f$ (which is equivalent to taking $\bar{\varepsilon}_0^f = 1.00$). In each case 100,000 initial failure strains were generated, and 200 "bins" (bars in the plots) were used across the range of strains plotted. The ordinate of each bar indicates the number of initial failure strains of the abscissa value that were generated (within a "$\Delta\varepsilon$" range equal to the thickness of a bar).[9] As mentioned, the realizations were generated using a median value of exactly 1. The medians calculated from the generated data ranged from 0.9876 to 1.0030 with an average error of 0.4%, acceptable accuracy for 100,000 samples.

Several observations can be made regarding the distributions in Fig. 2:

1. $m < 1$: The distribution has exponential characteristics (precisely exponential for $m = 1$).
2. $1 < m < \sim 3$: The distribution tails off to the right (positive skewness).
3. $m = \sim 3$: The distribution is nearly symmetrical (negligible skewness).
4. $m > \sim 3$: The distribution tails off to the left (negative skewness).
5. all m: No initial failure strains less than zero are obtained.[10]

Finally, observe that the distribution of initial failure strains becomes increasingly narrow as m increases. Extremely large values of m result in a fairly constant value of initial failure strain, approximately equal to $\bar{\varepsilon}_0^f$, the library value; i.e., approximately the same model as the standard (non-statistical) Johnson–Cook Fracture model. However, the decrease in the standard deviation with m is very slow. For the purposes of numerical simulation, a deterministic damage model would be appropriate only if the standard deviation in failure strain is below machine precision, and it can be shown[11] that this would require a Weibull modulus on the order of 10^{16}. Even the most uniform materials have a Weibull modulus orders of magnitude smaller than this. Hence, variation in failure strain should be included in any simulation to obtain realistic localization of failure.

In a graphical way, Fig. 3 shows the final observation. Here, the distributions are plotted as lines on a single abscissa scale and a single ordinate scale to facilitate comparison. Although the total number of samples remains 100,000 for each case, bin sizes used to plot Fig. 3 were different from the bin sizes used to plot Fig. 2 (100 bins over the range of strains plotted in Fig. 3, compared to the 200 bins for Fig. 2 plots), resulting in differences in the ordinate values in the two figures. The distributions get narrower as m increases,

[8] A popular conceptualization of the hazard function is the "bathtub curve" (shaped like the profile of a bathtub), initially decreasing, then relatively constant for a time, then increasing. Applied to human life, this is easy to grasp (in life, the "stress" is time, or age, not strain): initially the hazard function (or failure rate, say, deaths/1000 births) is high but decreasing with time (or age); this corresponds to infant mortality. Throughout most of life the failure rate is nearly constant, but in old age the failure rate begins to increase.

[9] Similar to the integral of the PDF, the area under the envelope of the bars represents probability of occurrence. For example, the area of the bar representing a particular initial failure strain divided by the area of all bars equals the probability of occurrence of that particular initial failure strain.

[10] The $m = 0.50$ case *appears* to show a number of strains less than zero because the plotting algorithm plotted zeros in a bar placed to the left of zero. These plotted points were actually small positive values in the data rounded to four decimal places for plotting (i.e., 0.0000) and therefore were plotted as zeros; no negative values were contained in the data.

[11] For the Weibull distribution, the ratio of standard deviation to median is approximately $(\pi/\sqrt{6}m)$ for large m. Hence, a distribution with a standard deviation smaller than round-off error (roughly 10^{-16}) requires an unrealistically large Weibull modulus.

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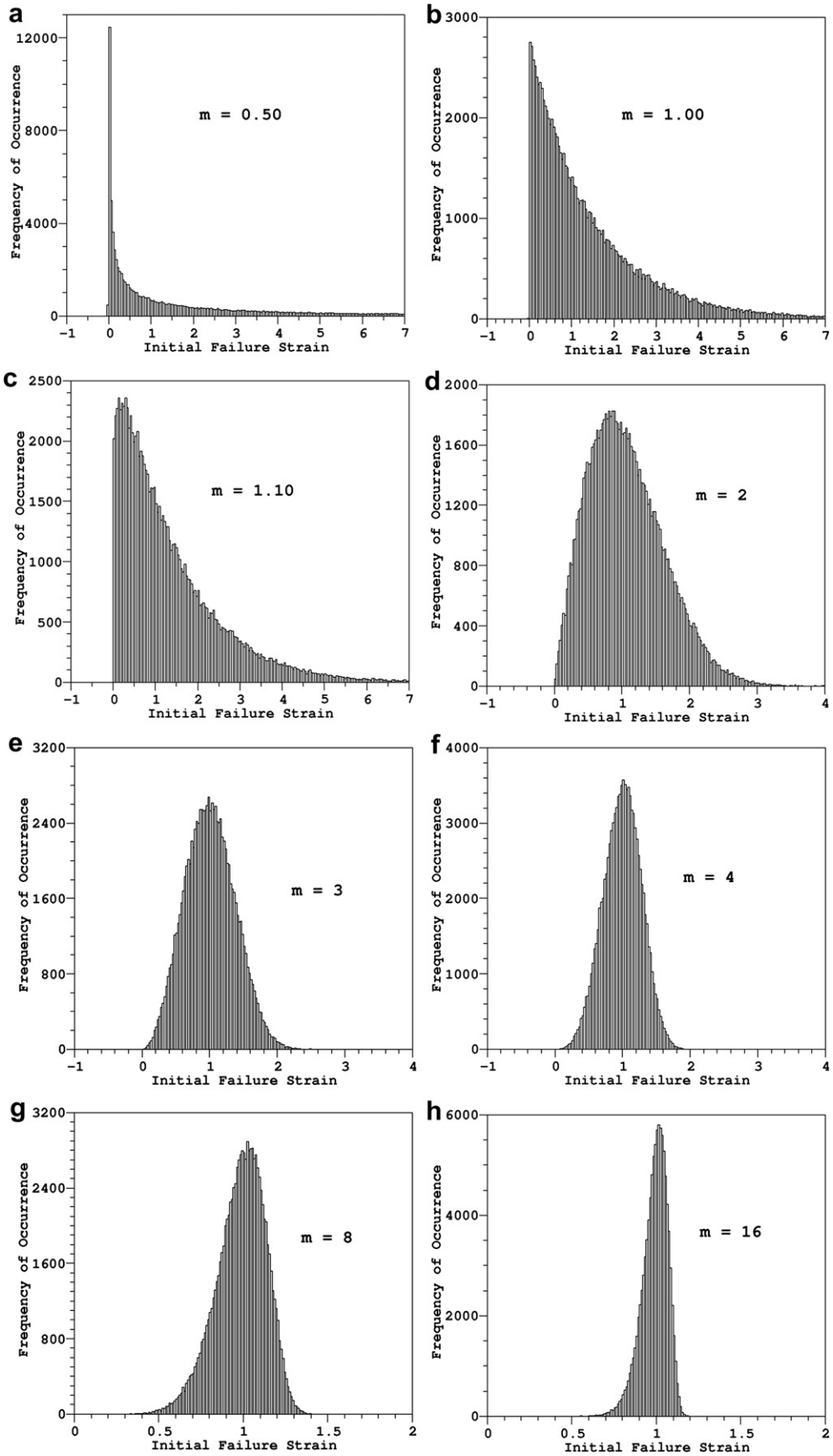

Fig. 2. Distributions of initial failure strains for various values of the Weibull modulus, *m* (scales vary).

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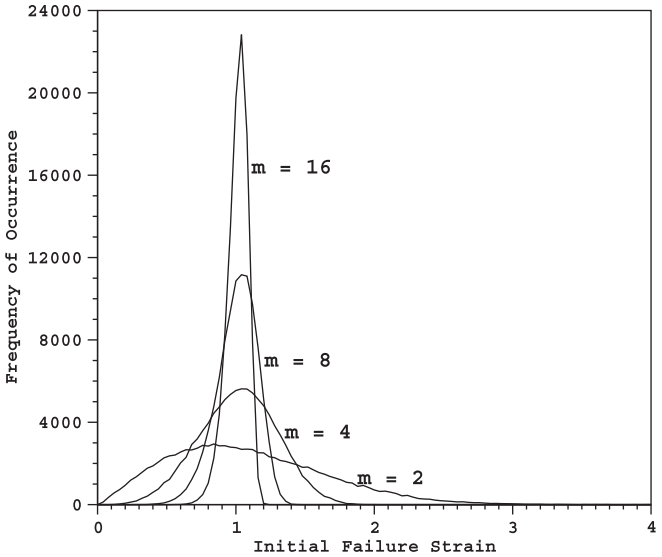

Fig. 3. Comparison of distributions of initial failure strains for various values of the Weibull modulus, *m*.

but the area under the curves is constant because the bin size and number of samples is constant.

Finally, the spatial randomness of the values can be inferred from Fig. 4, where the initial failure strains generated using $m = 2$ are plotted in the order in which they were generated, as indicated by the sequence number. Fig. 4a is a plot of all 100,000 values generated; Fig. 4b is an enlargement of the first 100 values generated. The sequence is apparently random. These values are inserted sequentially into the CTH computational elements by spatially regular I,J,K loops through the problem domain, so there is no resulting spatial regularity by which a body receives initial failure strains. However, as previously described (Fig. 2), a histogram of the data in Fig. 4 shows that there is a Weibull distribution of the frequency of occurrence of the various values of initial failure strain.

4. The size effect

Weibull mentioned "…the size effect on failures in solids…" in his statistical failure work [6]. This "size effect" has a venerable history. Griffith [10,11], in his classic crack theory work, conducted a series of experiments on glass fibers of varying diameters. He found that as fiber diameter decreased, fiber-breaking stress increased. In a homogeneous material, breaking *stress* does not depend on fiber diameter. Preceding even Griffith, da Vinci (as reviewed in [12]) found that short iron wires were stronger than long iron wires. In a homogeneous material, breaking strength does not depend on wire length. In both cases the larger samples have a higher probability of containing a critical flaw which will cause the specimen to fail.

This size effect must be applied to the current work, because the results of a simulation should be independent of the resolution (element size) used. Consider a finite, homogeneous specimen subjected to a uniform strain field; the probability that the specimen is safe from failure at a particular strain is P_s. If the specimen is divided (conceptually or computationally) into q elements, then (assuming independence) each element should have a safe-from-failure probability of Ψ_s, such that $\{\Psi_s\}^q = P_s$. If the specimen is divided into r elements, where $r > q$, then each element should have a safe-from-failure probability of Φ_s, such that $\{\Phi_s\}^r = P_s$.

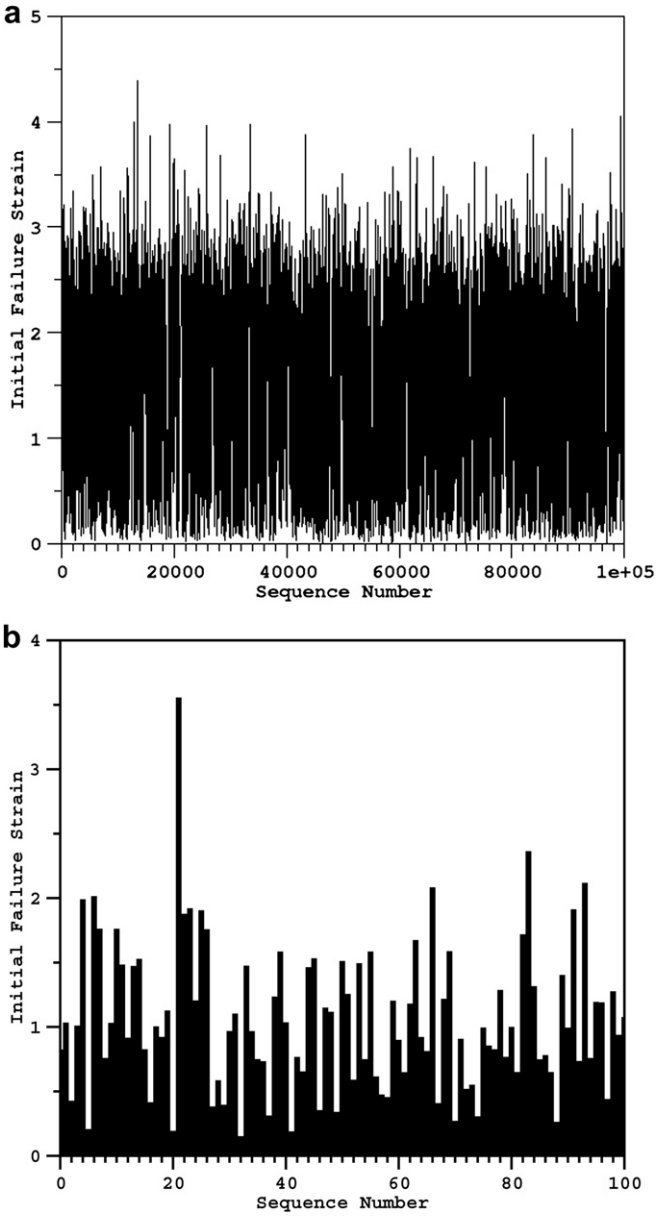

Fig. 4. Spatially random nature of the generated initial failure strains.

Hence $\{\Psi_s\}^q = \{\Phi_s\}^r$, and since Ψ_s and Φ_s are less than one and $r > q$, Φ_s must be greater than Ψ_s. In other words, smaller elements are more likely to survive a particular strain than large elements (but since there are more of them, the probability of specimen failure is the same). In the current context, the initial failure strain of a small element should be larger than that of a large element.

Eq. (12) describes the distribution of initial failure strains (ε_0^f) for a system of elements of an arbitrary size. The median value $\bar{\varepsilon}_0^f$ applies to some *fundamental* distribution of failure strains, a distribution for elements of a particular size, $V = \overline{V}$. If $V \neq \overline{V}$, the median value of the *size-corrected* distribution obtained by the statistical model (Eq. (12)) will be different from the median of the fundamental distribution (i.e., the library value $\bar{\varepsilon}_0^f$). The median of the size-corrected distribution is found by setting $R = ½$ in Eq. (12) to obtain

$$\bar{\varepsilon}_0^f = \bar{\varepsilon}_0^f \left(\frac{V}{\overline{V}}\right)^{-\frac{1}{m}}, \tag{17}$$

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where $\bar{\varepsilon}_0^f$ is used to differentiate the median of the size-corrected distribution from the median of the fundamental distribution, $\bar{\varepsilon}_0^f$. This formula for size effect has appeared in numerous publications on scale dependence [13]. The size-corrected distribution is the one that actually appears in a simulation, and the median of that distribution is offset from the median of the fundamental distribution by factor $(V/\overline{V})^{-1/m}$, which hereafter will be termed the "size effect factor". For simplicity, let

$$\alpha = \frac{V}{\overline{V}} \tag{18}$$

so that

$$\textit{size effect factor} = \alpha^{-1/m} \tag{19}$$

An increase in element volume results in a decrease of the median of the distribution of initial failure strains for any reference volume. Thus the size effect factor ensures that large elements are weaker on average than small elements (i.e., have smaller initial failure strains). However, the population of small elements is larger so that there is a greater chance that one of them will fail. The net result is that the probability of failure within a finite domain is unaffected by the mesh resolution used to cover that domain.

Some combinations of Weibull modulus with extremely small elements can cause a computational difficulty wherein the material is made artificially strong because the volume of the elements in a computation are small compared to $\overline{V}$. Johnson [14] reports the geometry of the specimens used in calibrating the fracture model (Eq. (1)). Three of the specimen geometries used by Johnson and Cook were analyzed to determine gage section volumes. The resulting volumes were 0.07 cm^3, 0.11 cm^3, and 0.31 cm^3. Comparing these to a finely resolved 3D computation, which may use an element volume of 0.001 cm^3, this is a difference of about two orders of magnitude; i.e., $\alpha \approx 0.01$.

The difficulty is shown graphically in Fig. 5, where Eq. (19) is plotted. In all cases, the size effect factor begins to grow without bound for small values of α (small element volumes). But problems occur long before the size effect factor becomes unbounded. For example, a ratio of volumes (α) of 0.01, combined with a Weibull modulus of $m = 2$ gives a size effect factor of more than 10. Failure strains for those elements will be (on average) 10 times the failure strains in the calibration experiment, possibly leading to a material unrealistically resistant to failure. Quantified examples of this issue will be given in the next section.

Fig. 6 shows another view of the same issue. For $\alpha = 0.01$, Weibull moduli less than 6 or so can be problematic. A ratio of volumes near unity causes no difficulty with any Weibull modulus.

Incidentally, the problem of size effects leading to extremely large shear failure strains in the current study does not occur in the alternative statistical perturbations explored by Brannon et al. [3] (see Fig. 1). With that approach, the hydrostatic tensile strength exhibits a size effect similar to that in Figs. 5 and 6, but the strength in shear can never become unbounded. In fact, the size dependence of shear strength in Brannon's approach is very similar in character to that reported for concrete [15] and other quasibrittle materials.

5. Application of the model: Perforation of an armor plate

Examples using different reference volumes and various values of the Weibull modulus are instructive. CTH was used to model a 1-inch-thick rolled homogeneous armor steel (RHA) plate impacted at normal incidence by a 30 mm tungsten APDS (Armor Piercing, Discarding Sabot) round at 1020 m/s. The Mie–Gruneisen equation of state and Johnson–Cook strength model were used to model both the penetrator and the target. CTH library values were used for the model constants. Johnson–Cook fracture with Weibull-distributed fracture constants was used to model the RHA; see Eqs. (2) and (3), with constants taken from the CTH library. The statistical population included at least 140,000 samples in all simulations. The tungsten penetrator was modeled with the standard (non-statistical) Johnson–Cook fracture model, using CTH library constants. At failure of the material in an element (as predicted by the Johnson–Cook fracture model), CTH inserts sufficient void into the element to relax the tensile pressure to zero. The simulations were fully 3D (i.e., symmetry was not employed), to take advantage of the *asymmetry* of the random placement of initial failure strains.

Fig. 7 shows the debris field behind the RHA plate at 250 μs after impact for various values of the Weibull modulus m. For the computations in Fig. 7, the reference volume is 0.100 cm^3, roughly the value of the gage sections for the Johnson–Cook

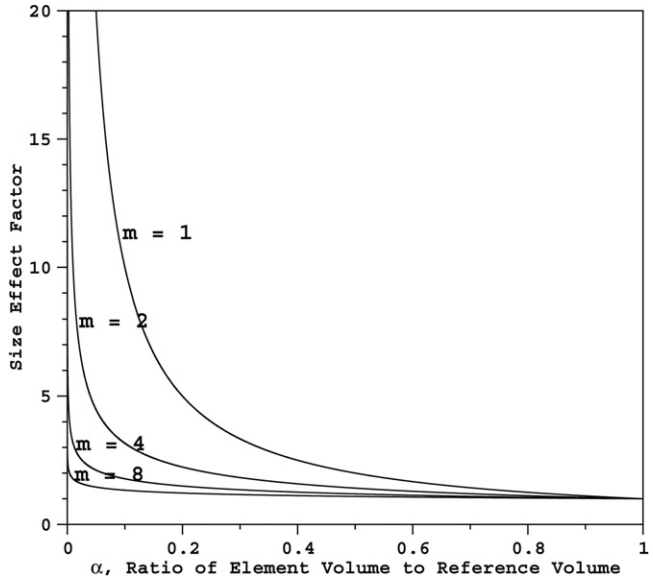

Fig. 5. The size effect factor under various conditions of volume ratio and Weibull modulus.

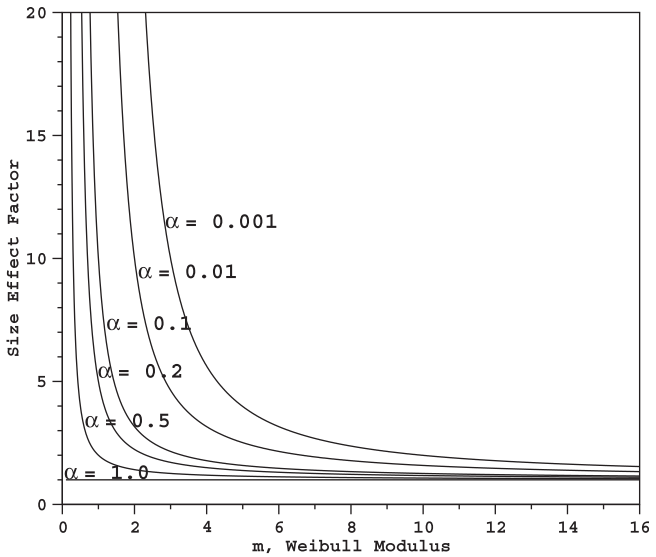

Fig. 6. The size effect factor for various volume ratios and Weibull moduli (is the volume ratio, $V/\overline{V}$).

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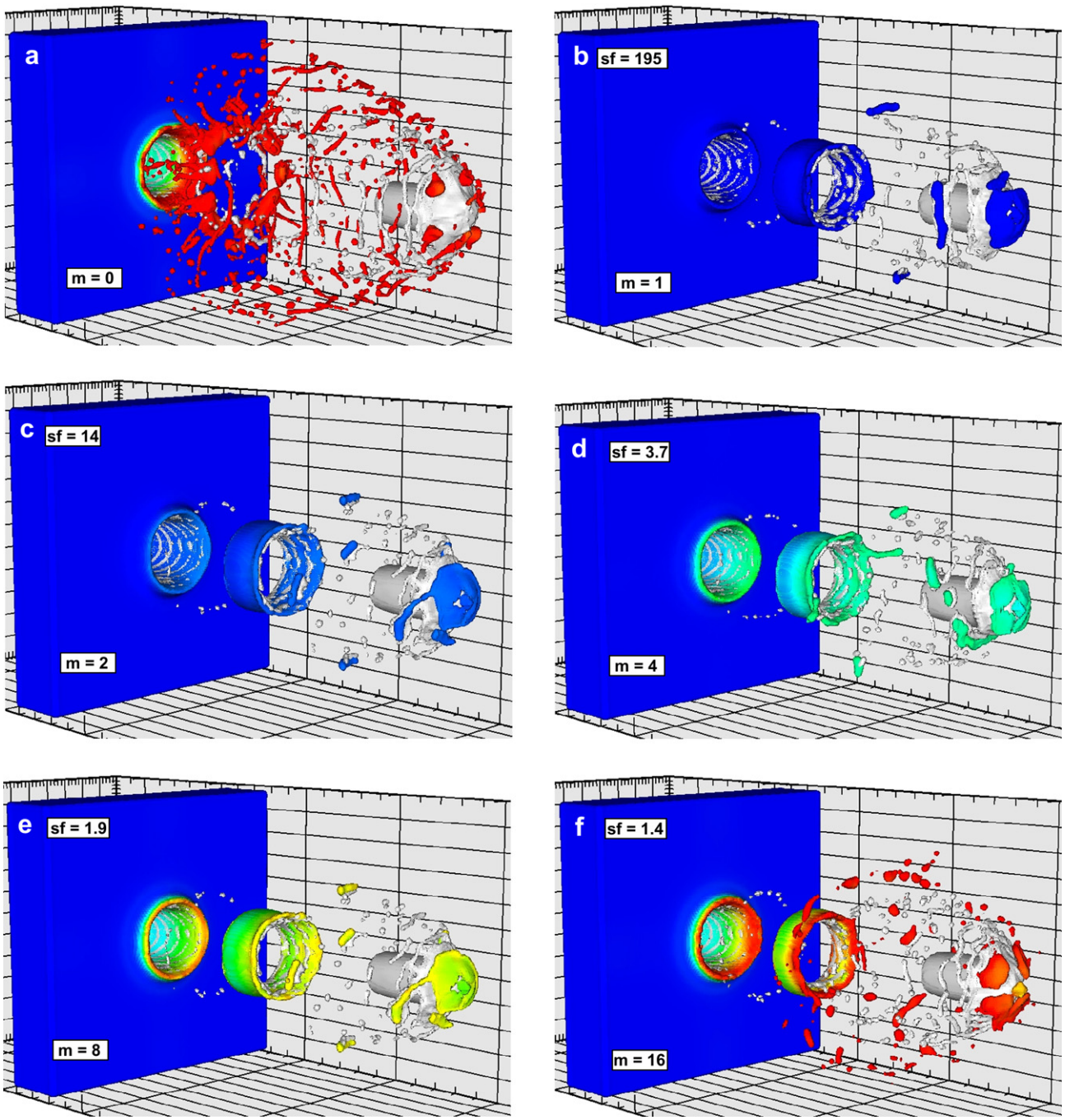

Fig. 7. Simulation results for a reference volume of 0.100 cm^3; sf is the size effect factor (Eq. (19)).

experiments. The grid is uniform, where each computational element is a cube, 0.08 cm on a side. Thus

$$\alpha = \frac{0.000512cm^3}{0.100cm^3} = 0.00512. \tag{20}$$

Damage as predicted by the Johnson–Cook Fracture model (JCF) is plotted on the RHA material: blue is no damage, red is fully damaged. The $m = 0$ case[12] shows the result for the standard JCF, where all computational elements contain identical initial failure strains (the narrowest distribution possible). There are many RHA fragments behind the plate, all of which are completely failed. For $m = 1$ the size effect factor is quite large, resulting in the material being artificially resistant to JCF failure.[13] The median initial failure strain for this case is unrealistically large, 270 cm/cm, resulting in

[12] A value of $m = 0$ is not permissible in the Weibull distribution function (Eq. (12)). Rather, a value of zero for the Weibull modulus is a flag for CTH, instructing it to use the standard Johnson–Cook model for each material so specified.

[13] CTH includes another failure criterion which is always active. This model predicts damage based on a user-supplied hydrostatic tensile pressure for undamaged material, called "PFRAC", and functions independently of JCF. In this case, a failed element is an element whose tensile pressure exceeds the user-supplied value. In the same way as previously described for JCF, tension is relaxed to zero by void insertion. The material will no longer support tension or shear. This is the mechanism causing the fragmentation of the JCF-undamaged material seen in Fig. 7.

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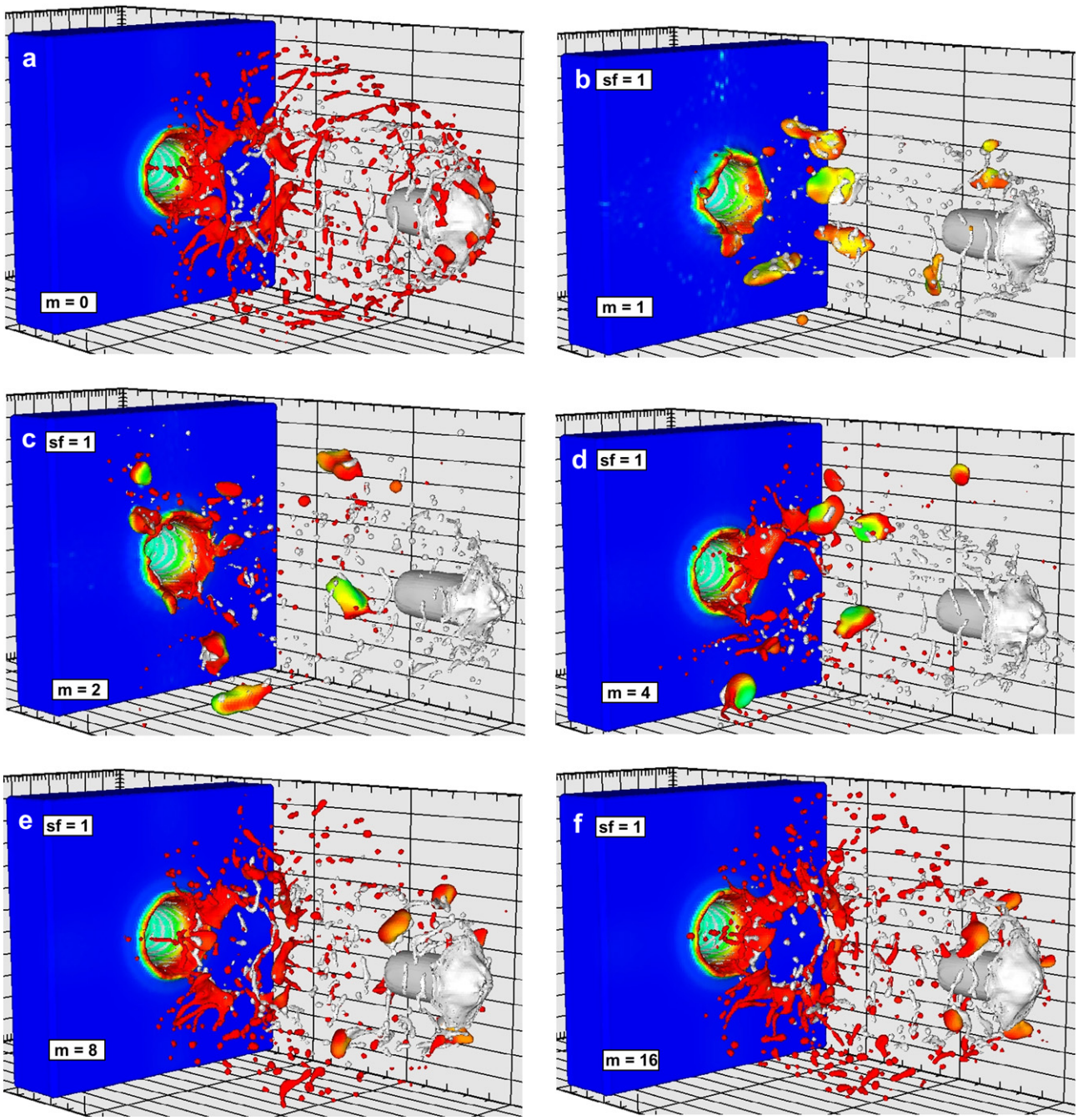

Fig. 8. Simulation results for a reference volume of 0.000512 cm^3; sf is the size effect factor (Eq. (19)).

no damage of the RHA. The debris field and lack of damage are unrealistic, at best. But as m increases, the small value of α is overcome by the exponent so that the size effect factor approaches unity (Eq. (19) and Fig. 6). The initial failure strain becomes more reasonable, and the result of the simulation appears more realistic. Consistent with expectations based on Fig. 6, for $a = 0.00512$ a value of $m = 8$ provides results that are not quite reasonable; $m = 16$ is required to provide reasonable results for the current choice of reference volume. So here, a size effect factor of less than about 1.4 is required.

If the reference volume is equal to the element volume (0.000512 cm^3 here), α is unity and the size effect factor is one for all values of m. The results for this case are quite different, as can be seen in Fig. 8. The *median* of the distribution of initial failure strains ($\tilde{\varepsilon}_0^f$) should be the same for all values of m if $\alpha = 1$ (Eq. (19)). The computed values are as constant as the random number generator and the finite sample size permit. The median value of $(D_1 + D_2)$ for RHA was taken as 1.3, and the medians of the generated populations are within 5%.

The results in Fig. 8 are more reasonable than those of Fig. 7. The debris field is made up of largely, if not completely, failed material. The debris fields for the Weibull cases are asymmetric, owing to the spatial asymmetry of the distribution of initial failure strains. The Weibull modulus plays a major role in the number of RHA fragments and in the resulting size distribution. For a size effect factor of one, the larger the Weibull modulus, the more like the (narrowly distributed) standard Johnson–Cook fracture case ($m = 0$) the result becomes. For the $m = 16$ case, the distribution of initial

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failure strains is narrowest of the Weibull-distributed cases examined (see Fig. 3), and the result is most similar to the result for the standard Johnson–Cook model.

6. Application of the model: predicting fragment distributions

The utility of the statistical fracture model is examined by comparing two CTH simulations to experiments. Conditions of the simulations and experiments are the same as previously discussed: a 1-inch-thick RHA plate impacted at normal incidence by a 30 mm APDS tungsten round at 1020 m/s. Perforation of the target by the penetrator results in target spall and penetrator fragmentation behind the armor plate, known collectively as Behind Armor Debris (BAD). The fragments in the experiment were captured in a water tank and later recovered and measured. Larger fragments that passed through the water tank (e.g., the residual penetrator) were captured in a subsequent stack of Celotex. Over 96% of the penetrator mass was recovered, and 75% (±15% principally due to the uncertainty of weight and volume measurements on a large target plate) of the mass removed from the RHA target plate was recovered.

Fig. 9 shows the frequency distribution of fragment mass for the experiments and simulations. The ordinate gives the number of fragments of BAD with mass greater than the abscissa value. The gray region shows the range of values obtained in two experiments. The lower line indicates the CTH prediction using the standard (non-statistical) Johnson–Cook fracture model. The upper line indicates the CTH prediction using the statistical model with a Weibull modulus of 2 for the penetrator and 8 for the target[14] ($\alpha = 1$ for both materials). Since the experiment did not distinguish between fragments from the RHA plate and fragments from the Tungsten penetrator, the simulation results present the total of both materials.

The graph shows that with the standard Johnson–Cook fracture model the number of very small fragments is over predicted, while the number of larger fragments is under predicted. Although the statistical Johnson–Cook fracture model over predicts the number small fragments as well[15], the important intermediate sizes are well predicted[16] (the few large-sized fragments are less likely to cause widespread damage). The moduli for the two materials were not optimized, so possibly a better fit could be obtained. Some adjustment in the median initial failure strains (ductility) and yield strengths also may lead to an improved match to the data.[17]

The plot of CTH results seen in Fig. 9 was made possible by software developed at the U.S. Army Research Laboratory (ARL) [16], called Frag Finder, which examines CTH output files and identifies all individual bodies in the computational domain at the computational cycle under study, and outputs the volume, mass, position vector and velocity vector for each. Without this tool, modeling Behind Armor Debris fields would be intractable.

[14] Little data presently exists for Weibull moduli for failure of these materials. The values used were selected because they fit the fragment population fairly well. The objective here was not to determine these moduli, but rather to show that CTH predictions could be improved by use of statistical fracture.

[15] Over prediction is not as severe for the statistical model: 295 total fragments vs. 346 for the standard model compared to 125 ± 25 fragments recovered in the two experiments. However, many very small fragments undoubtedly were not recovered.

[16] In simulations of recent experiments, both models accurately (within 5%) predicted the *total* mass of RHA fragments.

[17] Another variation in the simulation is possible in the present implementation of this statistical fracture model in CTH. The user has the ability to change the seeds for the random number generator (used in Eq. (12) to compute initial failure strain realizations), thus changing how the initial failure strains are spatially distributed (although the statistics would remain unchanged, as long as m remains unchanged). The effect of this variation on simulation results was not tested here.

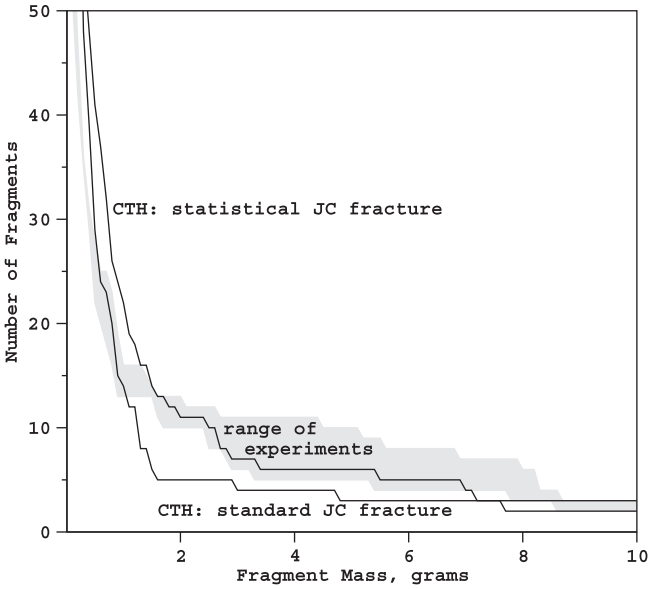

Fig. 9. Comparison of a statistical fracture calculation to a standard fracture calculation. The range of two experiments is shown in gray. For the statistical case, the Weibull modulus is 2 for the penetrator, and 8 for the target.

7. Further remarks on the size effect

The statistical fracture model as presented here is a work in progress. Some aspects of statistical fracture have not yet been considered, and the incompleteness is manifested in the issues with the size effect that have been discussed. Two important issues that still need to be addressed are interaction of flaws and correlated statistics.

The model as derived here, i.e., Eq. (12), is based on the assumption that flaws fail independently, as discussed previously (see Eq. (4)). But in reality, flaws in close proximity will affect each other. Interactions are varied, for example a crack in a plane normal to that of a growing crack may arrest the growth of the crack, while one in the same direction may facilitate its growth.

The second problematic assumption is that initiation of failure at a point will lead to cascading catastrophic failure everywhere. Recent on-going research indicates that geometrically random failure probabilities (as exist in the current model) do not provide cascading catastrophic failure. What may be going on at the microscale is correlated statistics, wherein the occurrence of a weak point in a material increases the probability that neighboring points also will be weak. This correlation is not included in the present model.

So the model as presented here is incomplete, but nevertheless is a first step in the right direction. Until the correct size effect functions (or correlated statistics) can be incorporated into the model, setting the reference volume equal to the element volume is reasonable. Of course this will make difficult the reproducing of size effects observed in the lab, and may introduce mesh-dependency into simulations. When the researcher is aware of these issues, the model (even in its present form) is still a very useful tool.

8. Conclusions

The following conclusions may be drawn from this study:

Including inherent variability in continuum mechanics simulations leads to more realistic predictions. For example, the statistical nature of behind armor debris experiments can now be modeled in CTH.

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Experimental data clearly show size effects in strength and failure but the proper way to incorporate them into a damage model remains a subject of research.

The present theory is based on statistics for the *onset* of failure whereas *progression* of failure undoubtedly requires either correlated statistics and/or revisions of the size effect to account for flaw interactions.

As with any computational damage model, further studies are warranted to ensure mesh independence of the predictions and accuracy in a variety of applications.

Acknowledgements

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