A
Numerical Integration

A.1 Introduction

The purpose of numerical integration, also known as *quadrature,* is to evaluate definite integrals of the type

$$A = \int_a^b f(x)\,dx \tag{A.1}$$

without using calculus. Quadrature gives only an approximate value for the integral, because calculus is the only method for performing the integration exactly. Numerical integration is useful in the following situations:

- The integration is difficult or tedious to perform analytically.
- The integral cannot be expressed in terms of known functions.
- The function $f(x)$ is unknown, but its values are known at discrete points.

Generally speaking, *integral* is a mathematical term for the sum of an infinite number of infinitesimal quantities. Consequently, the definite integral in Eq. (A.1) represents the summation of all the differential (infinitesimal) areas $dA = f(x)\,dx$ that lie between the limits $x = a$ and $x = b$, as seen in Fig. A.1. In numerical integration, the integral is approximated by adding the areas $A_1, A_2, A_3, \ldots, A_n$, of n (finite) panels, each of width Δx, as shown in Fig. A.2. Because the area of each panel must be estimated (integral calculus would be required to obtain the exact values), quadrature yields only an approximate value of the integral; that is,

$$A \approx \sum_{i=1}^{n} A_i \tag{A.2}$$

As a rule, a larger number of panels, with correspondingly smaller Δx, yields a more accurate result.

There are several methods available for estimating the areas of the panels. We discuss only the trapezoidal rule and Simpson's rule.

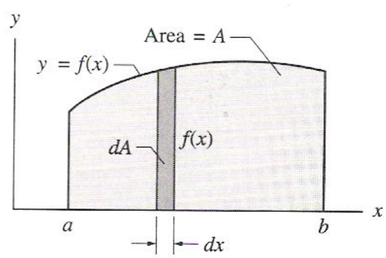

Fig. A.1

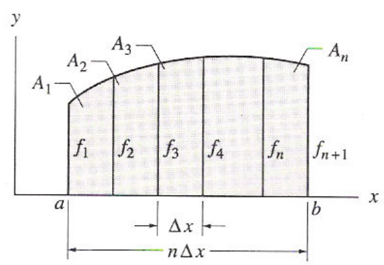

Fig. A.2

A.2 Trapezoidal Rule

In the trapezoidal rule, each panel is approximated by a trapezoid, as the name suggests. Recalling that the area of a trapezoid equals the base width times the average height, the areas of the panels in Fig. A.2 are $A_1 = [(f_1 + f_2)/2]\Delta x$,

$A_2 = [(f_2 + f_3)/2]\Delta x$, $A_3 = [(f_3 + f_4)/2]\Delta x, \ldots, A_n = [(f_n + f_{n+1})/2]\Delta x$. Here, we have introduced the notation $f_1 = f(a)$, $f_2 = f(a + \Delta x)$, $f_3 = f(a + 2\Delta x), \ldots, f_{n+1} = f(a + n\,\Delta x) = f(b)$.

Adding the areas of the panels, we have

$$A \approx \sum_{i=1}^{n} A_i = (f_1 + 2f_2 + 2f_3 + \cdots + 2f_n + f_{n+1})\Delta x/2 \quad \text{(A.3)}$$

Equation (A.3) is known as the *trapezoidal rule.*

The trapezoidal rule is sometimes written in the following form:

$$\boxed{A \approx \sum_{i=1}^{n+1} W_i f_i} \quad \text{(A.4)}$$

where the W_i are known as the *weights* and the expression $\Sigma_{i=1}^{n+1} W_i f_i$ is called the *weighted summation.* For the trapezoidal rule, the weights are

$$\boxed{\begin{aligned} W_1 &= W_{n+1} = \frac{\Delta x}{2} \\ W_i &= \Delta x \quad \text{for } 2 \leq i \leq n \end{aligned}} \quad \text{(A.5)}$$

A.3 Simpson's Rule

In the trapezoidal rule, the function $f(x)$ is approximated by a straight line within each panel of width Δx; that is, the curvature of $f(x)$ is neglected. This linearization may result in an unacceptably large error in the quadrature, particularly if the curvature of $f(x)$ is large and of the same sign throughout the interval $a \leq x \leq b$. Simpson's rule overcomes this deficiency by replacing the straight lines with parabolas. Because three points—that is, three values of $f(x)$—are required to define a parabola, Simpson's rule approximates the area of a pair of adjacent panels.

Referring again to Fig. A.2, it can be shown that if a parabola were passed through the three points (a, f_1), $(a + \Delta x, f_2)$, and $(a + 2\Delta x, f_3)$, the sum of the areas A_1 and A_2—that is, the area under the parabola—would be

$$A_1 + A_2 = (f_1 + 4f_2 + f_3)\,\Delta x/3$$

Similarly,

$$A_3 + A_4 = (f_3 + 4f_4 + f_5)\,\Delta x/3$$

$$A_5 + A_6 = (f_5 + 4f_6 + f_7)\,\Delta x/3$$

If these six panels were to represent the entire area, then the quadrature would yield

$$\begin{aligned} A &\approx A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \\ &= (f_1 + 4f_2 + 2f_3 + 4f_4 + 2f_5 + 4f_6 + f_7)\,\Delta x/3 \end{aligned}$$

Simpson's rule for n panels, where n must be an even number, becomes

$$\begin{aligned} A &\approx \sum_{i=1}^{n} A_i \\ &= (f_1 + 4f_2 + 2f_3 + 4f_4 + \cdots + 2f_{n-1} + 4f_n + f_{n+1})\Delta x/3 \end{aligned} \tag{A.6}$$

Introducing the concept of weights W_i, *Simpson's rule* can be written as

$$\boxed{A \approx \sum_{i=1}^{n+1} W_i f_i} \tag{A.7}$$

where the weights are

$$\boxed{\begin{aligned} W_1 &= W_{n+1} = \frac{\Delta x}{3} \\ &\left.\begin{aligned} W_i &= \frac{4\Delta x}{3} \quad i \text{ even} \\ W_i &= \frac{2\Delta x}{3} \quad i \text{ odd} \end{aligned}\right\} 2 \le i \le n \end{aligned}} \tag{A.8}$$

Because of its greater accuracy, Simpson's rule should be chosen over the trapezoidal rule. If the number of panels is odd, the area of one panel should be calculated using the trapezoidal rule, and then Simpson's rule can be used for the remaining panels.