Lab 05: TA Key for First Order Pre-Lab Exercise

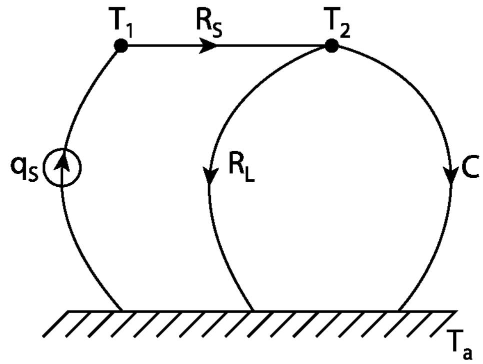

Figure 1: Linear graph of the first-order thermal system.

This is the linear graph for the thermal system for the first-order lab. The nodes T_1, T_2, and T_a are measured directly. The reference temperature, T_a is room temperature and is assumed to remain constant. The heat flux source, q_S, represents the heat transfer rate from the hot plate. T_2 is the temperature of the water measured by the temperature probe. The shaded equations indicate the solutions asked for in the pre-lab.

Part a)

Q: Derive the transfer function ($T_c(s)/Q_s(s)$) and time response ($T_c(t)$) for the first-order thermal system described in the introduction to this lab (assume a step input for $Q_s(s) = q_s/s$).

Part 1: The transfer function

Apply KCL at the T_2 node:

$$i_{R_s} = i_{R_L} + i_C \tag{1}$$

Substitute the following relationships:

$$i_{R_S} = q_S \tag{2}$$

$$i_{R_L} = \frac{T_2 - T_a}{R_L} \tag{3}$$

$$i_C = C\frac{d}{dt}\left(T_2 - T_a\right) \tag{4}$$

The result of this substitution is:

$$q_S = \frac{T_2 - T_a}{R_L} + C\frac{d}{dt}\left(T_2 - T_a\right) \tag{5}$$

For convenience, define $T_C = T_2 – T_a$ and simplify:

$$q_S = \frac{T_C}{R_L} + C\frac{d}{dt}\left(T_C\right) \tag{6}$$

Take the Laplace Transform of Equation 6:

$$Q_S(s) = \left(\frac{1}{R_L} + sC\right) T_C(s) \tag{7}$$

Rearranging the equation to find the transfer function $T_c(s)/Q_s(s)$:

$$\boxed{\frac{T_C(s)}{Q_S(s)} = \frac{1/C}{s + 1/R_L C}} \tag{8}$$

Part 2: The Time Response

Assuming the step input mentioned in the problem statement and substituting it into Equation 8 yields:

$$T_C(s) = \left(\frac{1/C}{s + 1/R_L C}\right)\frac{q_s}{s} \tag{9}$$

Use partial fraction expansion to separate Equation 9 into an expression that is a sum of two fractions that satisfy the following equation.

$$\frac{q_S/C}{s\left(s + 1/R_L C\right)} = \frac{A}{s} + \frac{B}{s + 1/R_L C} \tag{10}$$

Multiplying by the denominator of the right side of Equation 10 yields:

$$\frac{q_s}{C} = A\left(s + 1/R_L C\right) + Bs \tag{11}$$

For Equation 11 to be true the coefficients of s on both sides of the equation must be equal. This fact gives rise to the following set of simultaneous equations:

$$\begin{aligned} \frac{A}{R_L C} &= \frac{q_S}{C} \\ A + B &= 0 \end{aligned} \tag{12}$$

Solving for A and B yields:

$$\begin{aligned} A &= q_S R_L \\ B &= -q_S R_L \end{aligned} \tag{13}$$

Substituting this result into Equation 10 yields:

$$T_C(s) = q_s R_L\left(\frac{1}{s} + \frac{1}{s + 1/R_L C}\right) \tag{14}$$

Taking the inverse Laplace Transform of Equation 14 gives us the time response.

$$\boxed{T_C(t) = q_s R_L\left(1 - e^{-\left(1/R_L C\right)t}\right)} \tag{15}$$

Equation 15 reveals two important relations for the final temperature and the time constant (you will need these later).

$$(T_C)_f = q_S R_L \tag{16}$$

$$\tau = R_L C \tag{17}$$

Part b)

Q: Using this result, derive a symbolic expression for the heat flux source, q_s and the mass of the water, *m*.

The easiest way to determine the heat flux source is to use Equation 16 (assume the time response has reached steady state).

$$q_S = \frac{(T_C)_f}{R_L} \tag{18}$$

To determine a symbolic expression for the mass of the water you must use Equation 1 from the lab handout.

$$C = mC_p \tag{19}$$

where C_p is the specific heat of water. Combining Equations 17 and 19 gives us *m*.

$$m = \frac{\tau}{C_p R_L} \tag{20}$$

Part c)

Q: Using the result from Part a) above to express the time response of the system as a function of the measured temperatures T_2, and T_a as shown in the Linear graph of the system.

This result is almost trivial, but it will be important for when they are entering in the expression needed in the Matlab script that compares the predicted results to the data (lines 34 and 75 of `thermoLab.m`). Begin by substituting Equation 16 into Equation 15:

$$T_C(t) = (T_C)_f \left(1 - e^{-\left(1/R_L C\right)t}\right) \tag{21}$$

Now use $T_C = T_2 – T_a$ to find the solution.

$$T_2(t) = T_a + \left[(T_2)_f - T_a\right]\left(1 - e^{-\left(1/R_L C\right)t}\right) \tag{22}$$