ME 3210 Mechatronics II Laboratory

Lab 6: Second-Order Dynamic Response

Introduction

Second order differential equations approximate the dynamic response of many systems. In this lab you will model an aluminum bar as a second order mass-spring-damper system.

Second-Order Systems

This laboratory exercise focuses on a second-order mass-spring-damper system formed by a slender spring steel rod connected to a rotational inertia. The spring steel rod acts as a torsion spring and provides stiffness to the system via its material properties (Young's modulus); damping is provided by the air drag, and energy dissipation in the remainder of the system components. The generalized characteristic equation that describes every second-order system is

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^{\,2} y = f(t), \tag{1}$$

where y is the system output, ζ is the dimensionless damping ratio or damping coefficient, ω_n is the natural frequency in radians per second, and $f(t)$ is some forcing function. However, for this lab, the equation of motion that describes the mass-spring-damper system of the torsion spring is the following differential equation:

$$\ddot{\theta} + \frac{b}{J}\dot{\theta} + \frac{k}{J}\theta = 0, \tag{2}$$

where θ is the angular position of the rotational inertia, b is the damping coefficient, k is the stiffness of the system, and J is the rotational inertia of the system. The solution to Equation 2 is sometimes referred to as the free response of the system because the system response is not excited by an external force ($f(t)=0$). There are three different solutions to the free response: underdamped, overdamped, and critically damped solutions. Each depends on the value of ζ; if $\zeta < 1$ then the system is underdamped. For the case of the spring steel rod used in this lab, the damping forces that dissipate the vibrational energy of the system are small. Therefore we assume that the system is underdamped. The time response for a generalized underdamped system with non-zero initial conditions is shown below in Figure 1.

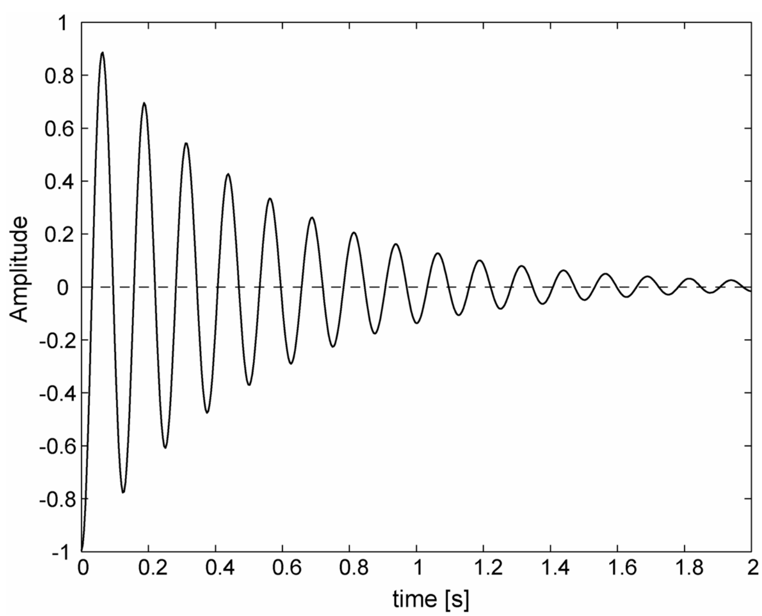

Figure 1: Generalized free response of an underdamped second-order system to non-zero initial conditions.

Determining ω_n and ζ Experimentally

One method for experimentally determining the parameters in Equation 1 is to solve the differential equation and perform some mathematical manipulation to isolate equations for specific, measurable parameters. Since the system is unforced, only the homogeneous solution is required. The homogeneous solution to Equation (1) is summarized in Equations (3) through (5).

$$\begin{aligned}\theta(t) &= \theta_0 e^{-\zeta\omega_n t}\left[\cos\left(\omega_d\ t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_d\ t\right)\right] \\ &= \frac{\theta_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos\left(\omega_d \mathrm{t} - \phi\right)\end{aligned} \tag{3}$$

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{4}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{5}$$

where θ_0 is the initial angular displacement of the end of the steel rod and $\theta(t)$ is the angular displacement as a function of time. This solution shows that a decaying exponential scales a sinusoidal oscillation, similar to the response in Figure 1. The time it takes to reach the first peak can be found by taking the time derivative of $\theta(t)$ and setting the result equal to zero.

$$\dot{\theta} = \left[\frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}\theta_0\cos\left(\omega_d t - \phi\right) + \frac{\omega_d}{\sqrt{1-\zeta^2}}\theta_0\sin\left(\omega_d t - \phi\right)\right]e^{-\zeta\omega_n t} = 0 \tag{6}$$

Since the exponential term theoretically never reaches zero, the time to the first peak can be determined through reduction using trigonometric identities and algebra. This procedure yields the equation for the time to peak, t_p, in terms of ω_n and ζ.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{7}$$

The time to peak is measured in seconds. If the system is sufficiently underdamped ($\zeta < 0.1$), then the term under the radical in Equation (7) approaches unity. Therefore we can estimate the natural frequency, ω_n, using the following relationship.

$$\omega_n \approx \frac{\pi}{t_p} \tag{8}$$

When a system is this underdamped the system response decays very slowly and it often takes a long time before the oscillations settle to a final or steady-state value. Generally, the settling time is defined as the time it takes for the system to enter a region around the system's final value without leaving. This bounded region is typically defined as a percentage of the steady-state value. For this experiment, the settling time will be defined to be when the system response settles to within 2% of its steady-state value. The settling time can be evaluated analytically as the time it takes the decaying exponential of Equation (3) to reach 0.02:

$$e^{-\zeta\omega_n t_s} = 0.02 \tag{9}$$

Solving Equation (9) for the settling time, t_s, yields the following relationship.

$$t_s = -\frac{\ln(0.02)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n} \tag{10}$$

So, now that we have expressions for the time to peak and the settling time, we can calculate ω_n and ζ using Equations (8) and (10) by examining the system response and measuring t_s and t_p.

Log Decrement Technique

Another way to calculate ω_n and ζ is to use the log decrement technique. This technique is exploits the periodicity of Equation (3) to determine ω_n and ζ. The time between successive peaks in the free response of the system is determined by the period, t_d, of the cosine function in Equation 2:

$$t_d = \frac{2\pi}{\omega_d}. \tag{11}$$

The amplitude of the first peak, which occurs at t_p, is defined as x_1. Substituting t_p and x_1 into Equation (3) yields the following equation:

$$x_1 = y_0 e^{-\zeta\omega_n t_p} \cos\left(\omega_d t_p - \phi\right). \tag{12}$$

The equation expressing the amplitude of the next peak is:

$$x_2 = y_0 e^{-\zeta\omega_n\left(t_p + t_d\right)} \cos\left(\omega_d\left(t_p + t_d\right) - \phi\right). \tag{13}$$

Subsequent expressions for amplitude peaks follow this pattern according to the Equation below:

$$x_{1+n} = y_0 e^{-\zeta\omega_n\left(t_p + nt_d\right)} \cos\left(\omega_d\left(t_p + nt_d\right) - \phi\right), \text{ for } n = 1, 2, 3\ldots. \tag{14}$$

The ratio of the initial peak to each successive peak is defined by the following expression:

$$\frac{x_1}{x_{1+n}} = e^{\zeta\omega_n n t_d}, \text{ for } n = 1, 2, 3\ldots. \tag{15}$$

Taking the natural logarithm of both sides of Equation (15) and substituting Equations (5) and (11) in for ω_d and t_d yields the following relationship:

$$\ln\left(\frac{x_1}{x_{1+n}}\right) = \zeta\omega_n n t_d = \frac{2\pi n\zeta}{\sqrt{1-\zeta^2}}, \text{ for } n = 1, 2, 3\ldots. \tag{16}$$

Once again we assume that for small damping ratios the term in the radical approaches unity, resulting in the following equation:

$$\delta(n) = \ln\left(\frac{x_1}{x_{1+n}}\right) = 2\pi n\zeta, \text{ for } n = 1, 2, 3\ldots. \tag{17}$$

This result shows that there is a linear relationship between n and $\delta(n)$ and that the damping ratio of the system contributes to the slope between the two. Therefore, the damping ratio can be determined through linear regression given values for n and $\delta(n)$. Once the damping ratio is determined through linear regression, the natural frequency, ω_n, can then be determined using the following equation (combine Equations (5) and (11)):

$$\omega_n = \frac{2\pi}{t_d\sqrt{1-\zeta^2}} \tag{18}$$

Experimental Procedure

1. Locate a spring-mass setup.

2. Connect the +12V port on the bench power supply to one of the lower two ports of the potentiometer and the -12V port to the other using banana cables.

3. Using a BNC cable and a BNC to banana adaptor, connect the wiper of the potentiometer to ACH0 of the DAQ terminal block (make sure that the black lead connects is grounded).

4. Using a BNC cable and a BNC to banana adaptor, connect the EXTREF to the ground of the bench power supply.

5. Open the 2_channel_oscope.vi from the LabView folder on the desktop and start running the VI by clicking the run button (single arrow) in the toolbar at the top of the screen.

6. Set the sampling rate and number of samples so that you will collect about five seconds of data, deflect the rotational inertia a small amount ($\theta_0 < 40°$), click the START button, and quickly release the end of the rotational inertia.

7. Examine the plot of the data and repeat Step 6 until you capture enough data to capture the moment when a 2% settling time is reached and you are satisfied with the response; save the data set with a .dat file extension to your disk.

8. Open the time (first column) and voltage (second column) data using Matlab or Excel and plot the data.

9. Calculate the natural frequency and damping ratio of the system response using Equations (8) and (10); record the values below:

   ζ = ____________________ ω_n = ____________________ rad/s

10. Determine ζ and ω_n using the log decrement technique.

    ζ = ____________________ ω_n = ____________________ rad/s

Questions

1. Is a second-order approximation sufficient to model this system? Why or why not?

2. Compare the results from Steps 9 and 10. In your opinion, which technique is more accurate? Why?

3. Determine the second-order pole locations for the system based on the results from either Step 9 or 10 (whichever you said was most accurate).

4. Briefly describe at least two other second-order systems.