Mechatronics II   Laboratory Exercise 5
Second Order Response

Theoretical Background

Second order differential equations approximate the dynamic response of many systems. The response of a generic second order system can be seen in Figure 1 at the bottom of the page. In this lab you will model an aluminum bar as a second order Mass-Spring-Damper system. The aluminum bar itself provides the mass, stiffness, and viscous damping for your model. The characteristic ODE that describes this system is of the form,

$$\ddot{y} + 2zw_n \dot{y} + w_n^2 y = 0 \tag{1}$$

where y is the system output, z is termed the damping ratio or damping coefficient (a dimensionless quantity), and w_n is the undamped natural frequency in rad/s. If $z < 1$, then the system is underdamped. This is the situation for the aluminum bars that will be used for this experiment.

One method for experimentally determining the parameters in Equation (1) comes from solving the differential equation and performing some mathematical manipulation. Since the system is unforced (thus the 0 on the right-hand side of equation (1)), only the homogeneous response needs to be solved:

$$y_h = y_0 e^{-zw_n t}\left[\cos\left(w_n \sqrt{1-z^2}\ t\right) + \frac{z}{\sqrt{1-z^2}} \sin\left(w_n \sqrt{1-z^2}\ t\right)\right]$$

$$= \left(\frac{1}{\sqrt{1-z^2}}\right) y_0 e^{-zw_n t} \cos\left(w_d t - f\right), \quad \text{where } f = \tan^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) \text{ and } w_d = w_n \sqrt{1-z^2} \tag{2}$$
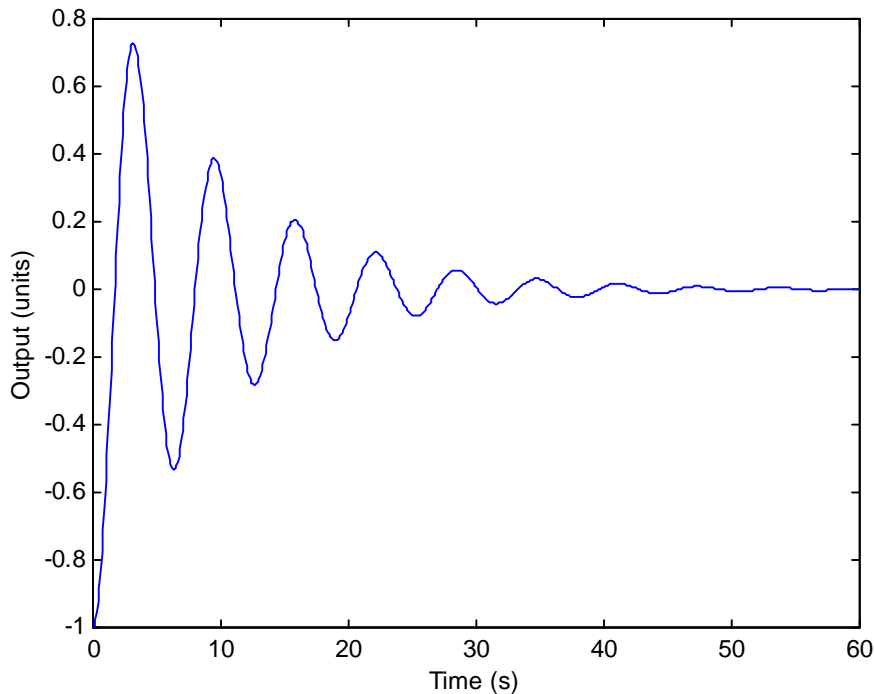

Figure 1: Time response of an underdamped second-order system

where y_0 is the initial condition of the output variable y. Inspection of this solution reveals a combination of a decaying exponential and sinusoidal oscillation, similar to the response in figure 1. The techniques presented here are based on measuring the amplitude of the peaks as well as when they occur. The time it takes to reach the first peak can be found by taking the time derivative of y_h and setting the result equal to zero, like so:

$$\dot{y} = \left[\frac{zw_n}{\sqrt{1-z^2}} y_0 \cos(w_d t - f) + \frac{w_d}{\sqrt{1-z^2}} y_0 \sin(w_d t - f)\right] e^{-zw_n t} = 0 \qquad (3)$$

Since the exponential term theoretically never reaches zero, the time to the first peak can be determined through reduction using trigonometric identities and algebra, resulting in

$$t_p = \frac{p}{w_n \sqrt{1-z^2}}$$
$$= \frac{p}{w_d}, \qquad (4)$$

where t_p is the time to peak in seconds. If the system is sufficiently underdamped ($z<0.1$), then the term under the radical in equation (4) approaches unity, allowing the natural frequency term to be solved according to the relation:

$$w_n \approx \frac{p}{t_p} \qquad (5)$$

For such an underdamped system, its takes some time for the response to settle to its final value. Generally, the *settling time* is defined as the time it takes for the system to enter a given bounded region about the system's final value without leaving. This bounded region is typically defined as a percentage of the final value. For this experiment, the settling time will be defined to be when the response settles to 2% of its final value (that is, 2% of the difference between the initial and final values). The settling time can be approximated analytically as the time it takes the decaying exponential to reach 0.02:

$$e^{-zw_n t_s} = 0.02 \qquad (6)$$

which can be solved for the settling time as follows:

$$t_s = -\frac{\ln(0.02)}{zw_n}$$
$$\approx \frac{4}{zw_n} \qquad (7)$$

for sufficiently underdamped ($z \lll 1$) systems. Using the preceding equations, the parameters z and w_n can be determined.

The second method you will use to characterize the system is the log decrement technique. The time between successive peaks in the response is determined to be

$$t_d = \frac{2p}{w_d} \qquad (8)$$

which is equivalently the period of the function $\sin(w_d - f)$. The amplitude of the first peak which occurs at t_p is defined as x_1. Substituted into equation (2), this becomes

$$x_1 = y_0 e^{-zw_n t_p} \cos\left(w_d t_p - f\right) \tag{9}$$

The amplitude of the next peak has the magnitude

$$x_2 = y_0 e^{-zw_n\left(t_p + t_d\right)} \cos\left(w_d\left(t_p + t_d\right) - f\right) \tag{10}$$

Subsequent amplitudes follow the pattern

$$x_{1+n} = y_0 e^{-zw_n\left(t_p + n t_d\right)} \cos\left(w_d\left(t_p + n t_d\right) - f\right) \tag{11}$$

where $n = 1, 2, 3, \ldots$ The ratio of two amplitudes is

$$\frac{x_1}{x_{1+n}} = e^{zw_n n t_d} \tag{12}$$

Taking the natural logarithm of both sides yields

$$\ln\left(\frac{x_1}{x_{1+n}}\right) = zw_n n t_d = \frac{2pnz}{\sqrt{1-z^2}} \tag{13}$$

As above, for small damping ratios, the term in the radical approaches unity, resulting in the relation

$$d(n) = 2pnz, \text{ where } d(n) = \ln\left(\frac{x_1}{x_{1+n}}\right) \tag{14}$$

If d versus n is plotted, a straight line through the data should result, whose slope is $2pz$. The natural frequency w_n is then found from the damped natural frequency w_d using equations (8) and (2).

Experiment
1. Secure an aluminum bar to the workbench using a large C-clamp.

2. Build an amplifier circuit. Verify that your circuit works and experimentally determine the gain of your amplifier using the bench potentiometer and a voltmeter. $k = $ _____.

3. Calibrate your system by displacing the tip and measuring the output voltage from your circuit. You must convert the output voltage to the deflection of the tip of the snow-aluminum bar for the post lab report. $d_{displacement} = $ _____, $V_{out} = $ _____.

4. Connect the output from your amplifier circuit to **AI_CH0** of the DAQ terminal block. Make sure to reference a common ground.

5. Run the 2NDORDER program from the CVI folder on the desktop.

6. Set the sampling rate and number of samples to the appropriate value and give the aluminum bar a step input by deflecting and releasing it while running the program. Be sure you hit the START button before you release the aluminum bar. Ensure the aluminum bar is sufficiently cantilevered. You may need to physically hold down the back of the aluminum bar securely to keep it from lifting off the bench.

7. Save your data when you get a response that decays sufficiently by the end of the data set. Ensure a 2% settling time is reached. Use Matlab or Excel to plot the data.

Questions

1. Determine z and w_n using equations (5) and (7).

$$z = \underline{\hspace{3cm}}, \quad w_n = \underline{\hspace{3cm}}.$$

2. Determine z and w_n using the log decrement technique (show your calculations here, but plot your data separately).

$$z = \underline{\hspace{3cm}}, \quad w_n = \underline{\hspace{3cm}}.$$

3. Compare the results from questions 1 and 2:

4. Is a second-order approximation sufficient to model this system? Why or why not?

5. Determine the second-order pole locations for the system based on the results from either question 1 or 2.

6. Briefly describe at least two other second-order systems.