Sliding Mode Control – Supplemental Derivation of Epsilon
ME EN 7200 – Nonlinear Controls – Spring 2003
4/15/03

7200 SUPPLEMENTAL NOTES

- FIND ε THAT GUARANTEES THAT THE ORIGIN IS ASYMPTOTICALLY STABLE WITH CONTINUOUS CONTROL.
  - ASSUME THAT $\theta_1 = \theta_2 = \frac{1}{2}$, $a = b = 1$, $\beta_0 = .1$, $k = 2$

$$\dot V = \frac{1}{2} x_1^2 = x_1 \dot x_1 = x_1 x_2 + \theta_1 x_1^2 \sin(x_2)$$

- LET $x_2 = s - kx_1$

$$\Rightarrow \dot V = x_1 s - kx_1^2 + \theta_1 x_1^2 \sin(x_2)$$

$$\le x_1 s - kx_1^2 + \theta x_1^2$$

$$\le -x_1^2(k - \theta) + x_1 s$$

$$\le -x_1^2(k-\theta) + |x_1|\varepsilon = -x_1^2 \frac{3}{2} + |x_1|\varepsilon$$

- SETTING $\dot V \le -\frac{1}{2} x_1^2$

$$-\frac{3}{2} x_1^2 + |x_1|\varepsilon \le -\frac{1}{2} x_1^2$$

$$\Rightarrow x_1^2 \ge |x_1|\varepsilon$$

$$\Rightarrow |x_1| \ge \varepsilon$$

$\Rightarrow$ TRAJECTORIES REACH THE SET Ω_ε IN FINITE TIME, WHERE,

$$\Omega_\varepsilon = \{ |x_1| \le |\varepsilon| , |s| < |\varepsilon| \}$$

- NOW WE EXAMINE STABILITY WITHIN THE INTERIOR OF Ω_ε

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INSIDE Ω_ε, $u = -x_1 - kx_2 + v$

$v = -\beta(x)\left(\frac{s}{\varepsilon}\right)$

- WE WILL PICK $V = \frac{1}{2} x_1^2 + \frac{1}{2} s^2$ AND FIND ε S.T. $\dot V < 0$

- FIRST, EXAMINE $\dot x_1$ & $\dot s$

$\dot x_1 = x_2 + \theta_1 x_1 \sin x_2$ BUT, $x_2 = s - kx_1$

$$\Rightarrow \dot x_1 = s - kx_1 + \theta_1 x_1 \sin(s - kx_1) = s - x_1(k - \theta \sin(\cdot))$$

$$\dot s = k\dot x_1 + \dot x_2$$

$$= kx_2 + k\theta_1 x_1 \sin(s-kx_1) + \theta x_2^2 + x_1 + \left(-x_1 - kx_2 - \beta(x)\frac{s}{\varepsilon}\right)$$

- RECALL THAT $\beta(x) = ka|x_1| + bx_2^2 + \beta_0$

$$\Rightarrow \dot s = k\theta_1 x_1 \sin(s - kx_1) + \theta_2 x_2^2 - (ka|x_1| + bx_2^2 + \beta_0)\frac{s}{\varepsilon}$$

$$= k\theta_1 x_1 \sin(s-kx_1) + x_2^2\left(\theta_2 - b\frac{s}{\varepsilon}\right) - ka|x_1|\frac{s}{\varepsilon} - \beta_0\frac{s}{\varepsilon}$$

- NOW PLUG INTO $\dot V = x_1 \dot x_1 + s\dot s$

$$\dot V = x_1(s - kx_1 + \theta_1 x_1 \sin(s-kx_1)) + s\left[k\theta_1 x_1 \sin(s-kx_1) + (s-kx_1)^2\left(\theta_2 - b\frac{s}{\varepsilon}\right) - ka|x_1|\frac{s}{\varepsilon} - \beta_0\frac{s}{\varepsilon}\right]$$

$$= -kx_1^2 + \theta_1 x_1^2 \sin(s-kx_1) + x_1 s + sk\theta_1 x_1 \sin(\,) + s(s-kx_1)^2\left(\theta_2 - b\frac{s}{\varepsilon}\right) - ka|x_1|\frac{s^2}{\varepsilon} - \beta_0\frac{s^2}{\varepsilon}$$

$$\le +x_1^2\left(-k + \theta_1 \overbrace{\sin(s-kx_1)}^{\le 1}\right) + |x_1||s| + k\theta_1|s||x_1| + |s|(s-kx_1)^2\left(\theta_2 - b\frac{s}{\varepsilon}\right) - ka|x_1|\frac{s^2}{\varepsilon} - \beta_0\frac{s^2}{\varepsilon}$$

$$\le -x_1^2(k-\theta_1) + |x_1||s|(1 + k\theta_1) + s(s-kx_1)^2\left(\theta_2 - b\frac{s}{\varepsilon}\right) - \beta_0\frac{s^2}{\varepsilon}$$

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NOTE THAT $(s - kx_1)^2 = s^2 - 2kx_1 + k^2 x_1^2$

– FURTHER, SUBSTITUTING FOR $k=2$, $a=b=1$, $\theta_1 = \theta_2 = 2$, $\beta_0 = .1$

$$\Rightarrow \dot{V} \leq -\frac{3}{2}x_1^2 + 2|x_1||s| + s\,(s^2 - 4x_1 + 4x_1^2)\left(\frac{1}{2} - \frac{s}{\varepsilon}\right) - .1\,\frac{s^2}{\varepsilon}$$

$$\leq -\frac{3}{2}x_1^2 + 2|x_1||s| + (s^3 - 4sx_1 + 4sx_1^2)\left(\frac{1}{2} - \frac{s}{\varepsilon}\right) - .1\,\frac{s^2}{\varepsilon}$$

$$\leq -\frac{3}{2}x_1^2 + 2|x_1||s| + \frac{1}{2}s^3 - 2sx_1 + 2sx_1^2$$

$$- \frac{s^4}{\varepsilon} + 4s^2\frac{x_1}{\varepsilon} + 4s^2\frac{x_1^2}{\varepsilon} \quad - .1\,\frac{s^2}{\varepsilon}$$

– NOT THAT $s^3 \leq s^2|\varepsilon|$

So,

$$\dot{V} \leq -\frac{3}{2}x_1^2 + 2sx_1^2 + 4x_1^2\frac{s^2}{\varepsilon} + 2|x_1||s| - 2sx_1$$

$$+ s^2\left[\frac{-.1}{\varepsilon} + 4\frac{x_1}{\varepsilon}\right] - \frac{s^4}{\varepsilon}$$

$$\leq \underbrace{x_1^2\left[-\frac{3}{2} + 2\varepsilon + 4\varepsilon\right]}_{-P_{11}} + \underbrace{2|x_1||s|}_{-2P_{12}} + \underbrace{s^2\left[\frac{-.1}{\varepsilon} + 4\right]}_{-P_{22}} - \frac{s^4}{\varepsilon}$$

$$\leq -\left[|x_1| \quad |s|\right]\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}}_{\text{WANT } P>0}\begin{bmatrix} |x_1| \\ |s| \end{bmatrix} - \frac{s^4}{\varepsilon}$$

NOTE THAT IF WE LET $\beta_0 = .5$, THEN $\varepsilon < .114$

$$\Rightarrow P_{11} > 0 \quad \& \quad P_{11}P_{12} - P_{12}^2 > 0$$

$$P_{11} = +\frac{3}{2} - 6\varepsilon > 0 \Rightarrow 6\varepsilon < \frac{3}{2} \Rightarrow \varepsilon < \frac{1}{4}$$

$$P_{11}P_{12} - P_{12}^2 = \left(+\frac{3}{2} - 6\varepsilon\right)\left(-4 + \frac{.1}{\varepsilon}\right) - 1^2 > 0$$

$$\left(\frac{3}{2} - 6\varepsilon\right)(.1 - 4\varepsilon) - \varepsilon = .15 - .6\varepsilon - 6\varepsilon - \varepsilon + 24\varepsilon^2 > 0$$

$\therefore$ PICK $\boxed{\varepsilon < .021}$ $\Rightarrow$ $0 < \varepsilon < .021$ or $.29 < \varepsilon < +\infty$