Derivation of the Energy Equation
ME 7710 – Environmental Fluid Dynamics
Spring 2013

This derivation follows closely from Bird, Stewart and Lightfoot (1960) but has been extended to include radiation and phase change. We can write the 1st law of thermodynamics for an open unsteady system shown in the figure below in words as follows:

$$\underbrace{\left\{\begin{array}{l}\text{rate of accumulation}\\ \text{of internal \& kinetic}\\ \text{energy}\end{array}\right\}}_{1} = \underbrace{\left\{\begin{array}{l}\text{rate of}\\ \text{internal \& kinetic}\\ \text{energy }\underline{\text{in}}\\ \text{by Advection}\end{array}\right\} - \left\{\begin{array}{l}\text{rate of}\\ \text{internal \& kinetic}\\ \text{energy }\underline{\text{out}}\\ \text{by Advection}\end{array}\right\}}_{2}$$

$$+\underbrace{\left\{\begin{array}{l}\text{net rate of}\\ \text{heat added by}\\ \text{Conduction}\end{array}\right\}}_{3} + \underbrace{\left\{\begin{array}{l}\text{net rate of}\\ \text{heat added by}\\ \text{Radiation}\end{array}\right\}}_{4} + \underbrace{\left\{\begin{array}{l}\text{net rate of}\\ \text{heat added by}\\ \text{Phase Change}\end{array}\right\}}_{5} - \underbrace{\left\{\begin{array}{l}\text{net rate of Work}\\ \text{done by the fluid}\\ \text{element on surroundings}\end{array}\right\}}_{6}$$

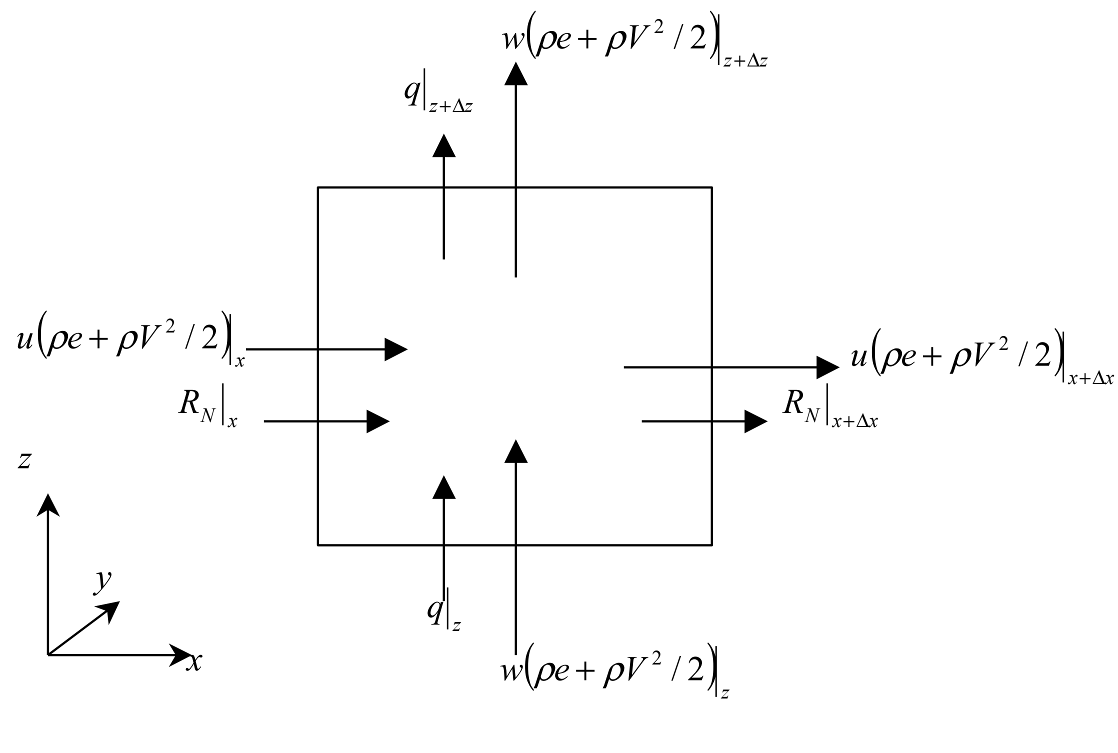

Figure 1: Schematic of a fluid element used for deriving the energy equation.

1. Rate of Accumulation of Internal (e) and Kinetic Energy ($V^2/2$) within the element:

$$\Delta x \Delta y \Delta z \frac{\partial}{\partial t}\left(\rho e + \rho V^2/2\right) \qquad (1)$$

2. Net rate of Advection of Internal and Kinetic Energy into the volume element:

$$\begin{aligned}&\Delta y \Delta z\left\{u\left(\rho e + \rho V^2/2\right)_{x+\Delta x} - u\left(\rho e + \rho V^2/2\right)_x\right\} \\ &+ \Delta x \Delta z\left\{v\left(\rho e + \rho V^2/2\right)_{y+\Delta x} - v\left(\rho e + \rho V^2/2\right)_y\right\} \\ &+ \Delta x \Delta y\left\{w\left(\rho e + \rho V^2/2\right)_{z+\Delta z} - w\left(\rho e + \rho V^2/2\right)_z\right\}\end{aligned} \qquad (2)$$

3. Net rate of Energy input by Conduction into the volume element (molecular):

$$\Delta y \Delta z\left(q_x\big|_x - q_x\big|_{x+\Delta x}\right) + \Delta x \Delta z\left(q_y\big|_y - q_y\big|_{y+\Delta y}\right) + \Delta x \Delta y\left(q_z\big|_z - q_z\big|_{z+\Delta z}\right) \qquad (3)$$

4. Net rate of Energy input by Radiation into the volume element:

$$\Delta y \Delta z\left(Rn_x\big|_x - Rn_x\big|_{x+\Delta x}\right) + \Delta x \Delta z\left(Rn_y\big|_y - Rn_y\big|_{y+\Delta y}\right) + \Delta x \Delta y\left(Rn_z\big|_z - Rn_z\big|_{z+\Delta z}\right) \qquad (4)$$

5. Net rate of Energy input by Phase Change into the volume element (Note that this is a Body Source term).

$$\Delta x \Delta y \Delta z\left(L_v E\right) \qquad (5)$$

Where L_v (2.45×10^6 J kg^{-1}) is the latent heat of vaporization or condensation and E (kg m^{-3} s^{-1}) is the evaporation rate or condensation rate per unit volume.

6. Net rate of Work done by the fluid element against the surroundings.

Recall, that work rate done by a force is the magnitude of the force multiplied by the velocity in the direction of the force, $\dot{W} = \vec{F} \cdot \vec{V}$, we will write the work rates as forces multiplied by velocities acting on our fluid element.

   a. *Work Against Body Forces* – Rate of doing work against the gravitational force

$$\Delta x \Delta y \Delta z\left(\rho u g_x + \rho v g_y + \rho w g_z\right) \qquad (6)$$

b. Work Against Surfaces Forces

i. Rate of Doing work against the **Pressure** at the six faces of a volume element

$$\Delta y \Delta z\left\{(pu)\big|_{x+\Delta x} - (pu)\big|_{x}\right\} + \Delta x \Delta z\left\{(pv)\big|_{y+\Delta y} - (pv)\big|_{y}\right\} + \Delta x \Delta y\left\{(pw)\big|_{z+\Delta z} - (pw)\big|_{z}\right\} \quad (7)$$

Note that here we are considering our work rate to be $(p \cdot \vec{n} dA) \cdot \vec{v}$ where $\vec{n}$ is the outwardly pointing normal vector from the fluid element.

ii. Rate of doing Work Against the **Viscous Forces**

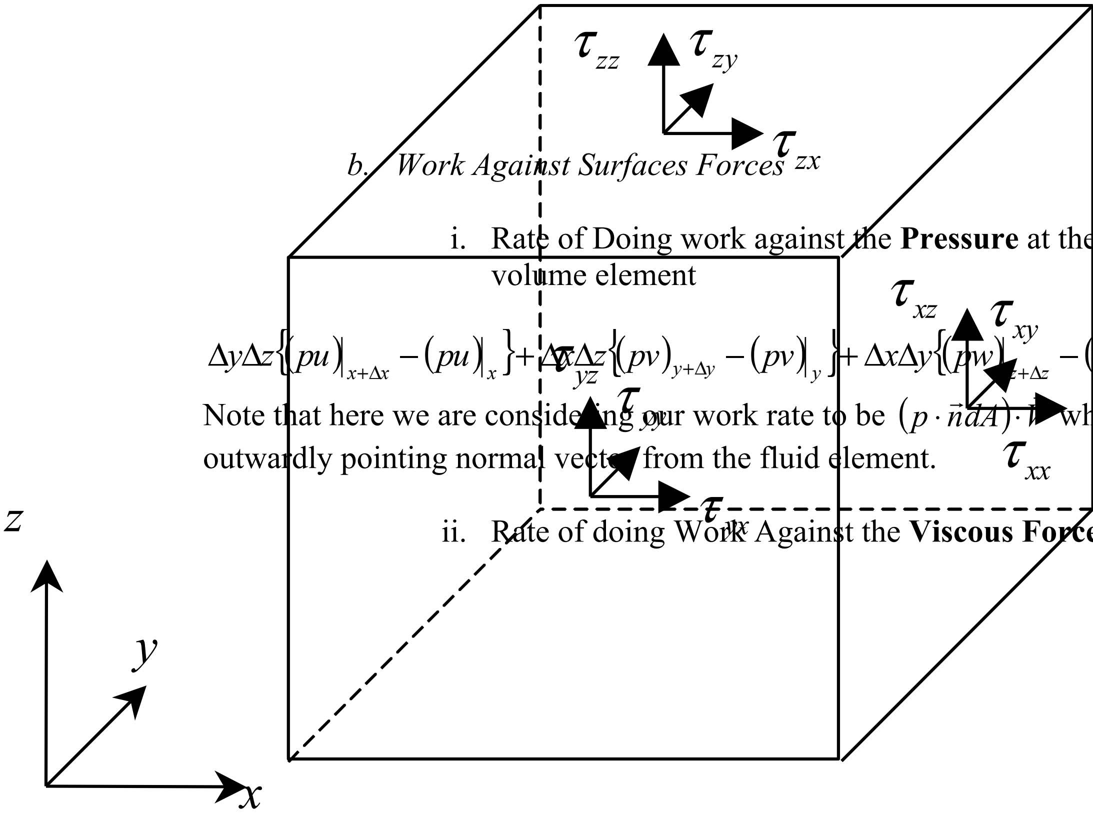

Figure 2: Fluid element volume indicating the stress sign convention for the derivation.

$$
\begin{aligned}
&\Delta y\Delta z\left\{\left(\tau_{xx}u+\tau_{xy}v+\tau_{xz}w\right)_x-\left(\tau_{xx}u+\tau_{xy}v+\tau_{xz}w\right)_{x+\Delta x}\right\}\\
&+\Delta x\Delta z\left\{\left(\tau_{yx}u+\tau_{yy}v+\tau_{yz}w\right)_y-\left(\tau_{yx}u+\tau_{yy}v+\tau_{yz}w\right)_{y+\Delta y}\right\}\\
&+\Delta x\Delta y\left\{\left(\tau_{zx}u+\tau_{zy}v+\tau_{zz}w\right)_z-\left(\tau_{zx}u+\tau_{zy}v+\tau_{zz}w\right)_{z+\Delta z}\right\}
\end{aligned}
\tag{8}
$$

Combining Eqs (1-8), dividing by $\Delta x\Delta y\Delta z$ and taking the limit as $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$ yields the complete energy equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho e+\rho V^2/2\right)=-\left(\frac{\partial}{\partial x}u\left(\rho e+\rho V^2/2\right)+\frac{\partial}{\partial y}v\left(\rho e+\rho V^2/2\right)+\frac{\partial}{\partial z}w\left(\rho e+\rho V^2/2\right)\right)\\
&-\left(\frac{\partial}{\partial x}q_x+\frac{\partial}{\partial y}q_y+\frac{\partial}{\partial z}q_z\right)-\left(\frac{\partial}{\partial x}pu+\frac{\partial}{\partial y}pv+\frac{\partial}{\partial z}pw\right)-\left(\frac{\partial}{\partial x}Rn_x+\frac{\partial}{\partial y}Rn_y+\frac{\partial}{\partial z}Rn_z\right)\\
&+\rho\left(ug_x+vg_y+wg_z\right)-L_pE\\
&-\left(\frac{\partial}{\partial x}\left(\tau_{xx}u+\tau_{yx}v+\tau_{zx}w\right)+\frac{\partial}{\partial y}\left(\tau_{xy}u+\tau_{yy}v+\tau_{zy}w\right)+\frac{\partial}{\partial z}\left(\tau_{xz}u+\tau_{yz}v+\tau_{zz}w\right)\right)
\end{aligned}
$$

As with the derivation of the momentum equation we now utilize Newtonian expressions to relate the velocity gradients and stresses, namely

$$
\begin{aligned}
\tau_{xx}&=-2\mu\frac{\partial u}{\partial x}+\frac{2}{3}\mu\left(\nabla\cdot\vec{V}\right) &\qquad \tau_{xy}&=\tau_{yx}=-\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\\
\tau_{yy}&=-2\mu\frac{\partial v}{\partial y}+\frac{2}{3}\mu\left(\nabla\cdot\vec{V}\right) &\qquad \tau_{xz}&=\tau_{zx}=-\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\\
\tau_{zz}&=-2\mu\frac{\partial w}{\partial z}+\frac{2}{3}\mu\left(\nabla\cdot\vec{V}\right) &\qquad \tau_{zy}&=\tau_{yz}=-\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)
\end{aligned}
$$

or more compactly this may be written,

$$
\tau_{ij}=-2\mu S_{ij}+\frac{2}{3}\mu\left(\nabla\cdot\vec{V}\right)\delta_{ij}
$$

We now subtract the Kinetic Energy equation (derived early in the course) from the complete energy equation to yield the Thermal Energy Equation:

$$\underbrace{\rho \frac{De}{Dt}}_{I} = \underbrace{-\nabla \cdot \vec{q}}_{II} - \underbrace{p\left(\nabla \cdot \vec{V}\right)}_{III} - \underbrace{\nabla \cdot \vec{R}_n}_{IV} + \underbrace{L_v E}_{V} + \underbrace{\mu \Phi_v}_{VI} \tag{9}$$

Term	Physical interpretation of each term
I	**Rate of Gain** of internal energy per unit volume
II	Rate of internal energy input by **Conduction** per unit volume
III	Reversible rate of internal energy increase per unit volume by **Compression**
IV	Rate of internal energy input by **Net Radiation** per unit volume
V	Rate of internal energy input by **Phase Change**
VI	Irreversible rate of internal energy increase per unit volume by **Viscous Dissipation**

The viscous term written out in full is

$$\Phi_v = -2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2$$

$$+\frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

or,

$$\Phi_v = -2\mu S_{ij} S_{ij} + \frac{2}{3}\mu\left(\nabla \cdot \vec{V}\right)^2.$$

We would now like to express the Equation of Thermal Energy in terms of temperature and heat capacity rather than internal energy:

Recall from thermodynamics, that $e = e(\alpha, T)$, where α is the specific volume and *T* the absolute temperature. Thus,

$$de = \left(\frac{\partial e}{\partial \alpha}\right)_T d\alpha + \left(\frac{\partial e}{\partial T}\right)_\alpha dT$$

multiplying by the density and considering the substantial derivatives

$$\rho \frac{De}{Dt} = \left(\frac{\partial e}{\partial \alpha}\right)_T \rho \frac{D\alpha}{Dt} + \rho C_v \frac{DT}{Dt}$$

where C_v is the specific heat of the fluid at constant volume. Writing the specific volume as the inverse of the density and using the product rule of calculus,

$$\rho \frac{D\alpha}{Dt} = \rho \frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\rho}\nabla \cdot \vec{V}$$

Then for incompressible flow where $\nabla \cdot \vec{V} = 0$, we have

$$\rho C_v \frac{DT}{Dt} = -\nabla \cdot \vec{q} - \nabla \cdot \vec{R}_n + L_v E + \mu \Phi'_v \qquad (10)$$

where the viscous dissipation term simplifies to

$$\Phi_v = -2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2$$

or

$$\Phi_v = -2\mu S_{ij} S_{ij}$$

for incompressible flow.

Using <u>Fourier's Law of Heat Conduction</u> allows us to write term *II* of Eq. (9) in terms of temperature,

$$q_x = -k\frac{dT}{dx}$$

$$\vec{q} = -k\nabla T$$

where k is the thermal conductivity of the fluid. Thus,

$$-\nabla \cdot \vec{q} = k\nabla \cdot \nabla T = k\nabla^2 T$$

Substituting in Eq. (10) gives us an equation for the change in temperature,

$$\rho C_v \frac{DT}{Dt} = k\nabla^2 T - \nabla \cdot \vec{R}_n - L_v E + \mu \Phi'_v \qquad (11)$$

This equation yields temperature changes from:

1. Heat conduction
2. Radiation divergence
3. Phase change
4. Viscous heating

For a constant pressure fluid we can make the following substitution

$$de = -p\,d\alpha + C_p\,dT$$

which for an incompressible fluid leads to

$$\rho C_p \frac{DT}{Dt} = k\nabla^2 T - \nabla \cdot \vec{R}_n + L_v E + \mu \Phi'_v$$

which is essentially stating that we can switch C_v and C_p. This justified when the pressure terms are neglected in a gas flow energy equation. What remains is approximately an enthalpy change.

$$\frac{DT}{Dt} = \frac{k}{\rho C_p}\nabla^2 T - \frac{1}{\rho C_p}\nabla \cdot \vec{R}_n + \frac{L_v E}{\rho C_p} + \frac{\mu \Phi'_v}{\rho C_p}$$

If viscous heating is small and we define thermal diffusivity as $K = \dfrac{k}{\rho C_p}$

$$\boxed{\frac{DT}{Dt} = K\nabla^2 T - \frac{1}{\rho C_p}\nabla \cdot \vec{R}_n + \frac{L_v E}{\rho C_p}}$$