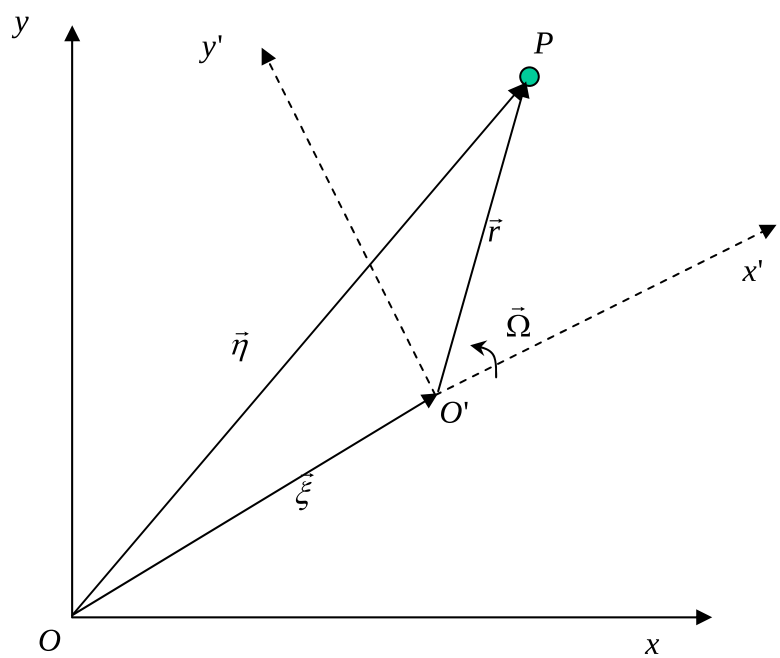
Derivation of the Momentum Equation in a non-inertial Coordinate System
ME 7710 – Environmental Fluid Dynamics
Spring 2007



Consider the inertial coordinate system *xyz* with origin *O* and the non-inertial coordinate system $x'y'z'$ with origin O'. The $x'y'z'$ coordinate system is allowed to translate and rotate at an angular velocity $\vec{\Omega}$. A particle *P* can be identified with respect to the non-inertial coordinate system by the position vector $\vec{r}$ and with respect to the inertial coordinate system by the position vector $\vec{\eta}$. The position vector $\vec{r}$ can be written in terms of the non-inertial coordinate system as,

$\vec{r} = x'\hat{i} + y'\hat{j} + z'\hat{k}$,

and the position vector of *P* in the inertial frame is related to the non-inertial position vector by

$$\vec{\eta} = \vec{\xi} + \vec{r} \ . \qquad (1)$$

The velocity of with respect to the inertial coordinate system is obtained by differentiating (1) with respect to time.

$$\vec{V}_{xyz} = \frac{d\vec{\eta}}{dt} = \frac{d\vec{\xi}}{dt} + \frac{d\vec{r}}{dt} = \vec{V}_{O'} + \frac{d\vec{r}}{dt} \qquad (2)$$

where, $\vec{V}_{O'}$ is the velocity of the non-inertial frame origin. The unit vectors $\hat{i}$ and $\hat{j}$ that describe $\vec{r}$ are not constant (although there magnitude is) hence, differentiation of $\vec{r}$ must include differentiating the unit vectors with respect to time also. Doing this yields,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x'\hat{i} + y'\hat{j} + z'\hat{k}\right) = \frac{dx'}{dt}\hat{i} + x'\frac{d\hat{i}}{dt} + \frac{dy'}{dt}\hat{j} + y'\frac{d\hat{j}}{dt} + \frac{dz'}{dt}\hat{k} + z'\frac{d\hat{k}}{dt}$$

Recall from the discussion in class of the centrifugal force that for a constant magnitude vector $\vec{R}$ rotating: $\frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}$. Hence,

$$x'\frac{d\hat{i}}{dt} = x'\left(\vec{\Omega} \times \hat{i}\right) \tag{3}$$

and

$$y'\frac{d\hat{j}}{dt} = y'\left(\vec{\Omega} \times \hat{j}\right) \tag{4}$$

and

$$z'\frac{d\hat{k}}{dt} = z'\left(\vec{\Omega} \times \hat{k}\right) \tag{5}$$

The velocity with respect to the rotating frame is

$$\vec{V}_{x'y'z'} = \frac{dx'}{dt}\hat{i} + \frac{dy'}{dt}\hat{j} + \frac{dz'}{dt}\hat{k} \tag{6}$$

Substituting into (3)(4)(5)(6) into (2) gives

$$\frac{d\vec{r}}{dt} = \frac{dx'}{dt}\hat{i} + \frac{dy'}{dt}\hat{j} + \frac{dz'}{dt}\hat{k} + x'\left(\vec{\Omega} \times \hat{i}\right) + y'\left(\vec{\Omega} \times \hat{j}\right) + z'\left(\vec{\Omega} \times \hat{k}\right)$$

or rewriting the cross product terms on the right hand side

$$\frac{d\vec{r}}{dt} = \vec{V}_{x'y'z'} + \vec{\Omega} \times \left(x'\hat{i} + y'\hat{j} + z'\hat{k}\right)$$

and substituting $\vec{r} = x'\hat{i} + y'\hat{j} + z'\hat{k}$ yields

$$\boxed{\frac{d\vec{r}}{dt} = \vec{V}_{x'y'z'} + \vec{\Omega} \times \vec{r}\ .} \tag{7}$$

Substituting (7) into (2) gives the following equation for the velocity in the inertial frame:

$$\vec{V}_{xyz} = \vec{V}_{O'} + \vec{V}_{x'y'z'} + \vec{\Omega} \times \vec{r}\ , \tag{8}$$

In words, (8) states:

velocity of *P* in inertial frame *xyz* = Velocity or Origin O' + Velocity of P in $x'y'z'$ + velocity term associated with rotating coordinate system.

Differentiating (8) with respect to time yields an equation for the acceleration in the inertial reference frame *xyz*, namely

$$\vec{a}_{xyz} = \frac{d\vec{V}_{xyz}}{dt} = \frac{d}{dt}\left(\vec{V}_{O'} + \vec{V}_{x'y'z'} + \vec{\Omega}\times\vec{r}\right) = \frac{d\vec{V}_{O'}}{dt} + \frac{d\vec{V}_{x'y'z'}}{dt} + \frac{d\left(\vec{\Omega}\times\vec{r}\right)}{dt}$$

$$\vec{a}_{xyz} = \vec{a}_{O'} + \frac{d\vec{V}_{x'y'z'}}{dt} + \frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt} \qquad (9)$$

Recall, that $\vec{r}$ and $\vec{V}_{x'y'z'}$ are measured with respect to the rotating frame. Next, (7) is differentiated (again giving consideration to the unit vectors) to give

$$\frac{d\vec{V}_{x'y'z'}}{dt} = \vec{a}_{x'y'z'} + \vec{\Omega}\times\vec{V}_{x'y'z'}\,. \qquad (10)$$

Substituting (10) into (9) yields

$$\vec{a}_{xyz} = \vec{a}_{O'} + \vec{a}_{x'y'z'} + \vec{\Omega}\times\vec{V}_{x'y'z'} + \frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt} \qquad (11)$$

Again substituting equation (7) into the last term in (11) gives,

$$\vec{\Omega}\times\frac{d\vec{r}}{dt} = \vec{\Omega}\times\left(\vec{V}_{x'y'z'} + \vec{\Omega}\times\vec{r}\right) = \vec{\Omega}\times\vec{V}_{x'y'z'} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) \qquad (12)$$

and substituting (12) into (11) yields,

$$\vec{a}_{xyz} = \vec{a}_{O'} + \vec{a}_{x'y'z'} + \vec{\Omega}\times\vec{V}_{x'y'z'} + \frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\vec{V}_{x'y'z'} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$$

$$\boxed{\underbrace{\vec{a}_{xyz}}_{I} = \underbrace{\vec{a}_{O'}}_{II} + \underbrace{\vec{a}_{x'y'z'}}_{III} + \underbrace{\frac{d\vec{\Omega}}{dt}\times\vec{r}}_{IV} + \underbrace{2\vec{\Omega}\times\vec{V}_{x'y'z'}}_{V} + \underbrace{\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)}_{VI}} \qquad (13)$$

Term	Physical interpretation of each term
I	Absolute rectilinear acceleration of the particle *P* relative to the inertial coordinates system xyz.
II	Absolute rectilinear acceleration of the non-inertial reference frame ($x'y'z'$) relative to the inertial coordinates system xyz. (This will be zero for the earth)
III	Rectilinear acceleration as measured in the non-inertial reference frame ($x'y'z'$) .
IV	Tangential acceleration due to angular acceleration of the moving reference frame. (This will also be zero for the earth)
V	Coriolis acceleration due to motion of a particle in the rotating frame.
VI	Centripetal acceleration due to rotation of the moving fram

For the rotating Earth, (13) simplifies to

$$\vec{a}_{xyz} = \vec{a}_{x'y'z'} + 2\vec{\Omega}\times\vec{V}_{x'y'z'} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) \qquad (14)$$

The Centrifugal Force and Newtonian Gravity ($\vec{g}_a$) are usually combined as:

$$\vec{g} = \vec{g}_a - \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) \qquad (15)$$

We will utilize this when we finish deriving the rest of the momentum equation.

The Coriolis Force: $2\vec{\Omega}\times\vec{V}_{x'y'z'}$

- Results in a curved path in a direction opposite to the direction of coordinate rotation.
- Acts perpendicular to the velocity vector.
- Can only change the direction of travel.

From the derivation given in class, the momentum equation given in an inertial reference frame is

$$\left(\frac{D\vec{V}}{Dt}\right)_I = -\frac{1}{\rho}\nabla\vec{P} + \nu\nabla^2\vec{V} + \vec{g}_a \qquad (16)$$

Substituting (14) into (16) yields

$$\left(\frac{D\vec{V}}{Dt}\right)_R + 2\vec{\Omega}\times\vec{V}_R + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) = -\frac{1}{\rho}\nabla\vec{P} + \nu\nabla^2\vec{V}_R + \vec{g}_a$$

or

$$\left(\frac{D\vec{V}}{Dt}\right)_R = -\frac{1}{\rho}\nabla\vec{P} - 2\vec{\Omega}\times\vec{V}_R + \nu\nabla^2\vec{V}_R + \left(\vec{g}_a - \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right) \qquad (17)$$

Substituting (15) into (17) and dropping the *R* yields the momentum equation for velocities measured on the rotating Earth.

$$\boxed{\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla\vec{P} - 2\vec{\Omega}\times\vec{V} + \nu\nabla^2\vec{V} + \vec{g}}$$

Note that these notes are based on the material from R.S. Azad’s “The Atmospheric Boundary Layer for Engineers,” Klewer (1993).