LU Decomposition Example

- Solve the following system of Algebraic Eqns

$$\begin{bmatrix} 4 & 3 & 1 \\ 3 & -1 & 5 \\ 2 & 4 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

STEP 1: Decompose $[A] \to [L] + [U]$, use Gauss-Elimination

(i) multiply (1) by $\frac{3}{4}$ and subtract from (2), replace (2) with result, call it (2')

(ii) multiply (1) by $\frac{2}{4}$ and subtract from (3), replace (3) with result, call it (3')

$$\begin{bmatrix} 4 & 3 & 1 \\ 3 & -1 & 5 \\ 2 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 3 & 1 \\ 0 & -\frac{13}{4} & \frac{17}{4} \\ 0 & \frac{5}{2} & 3/2 \end{bmatrix} \begin{matrix} (1) \\ (2') \\ (3') \end{matrix}$$

(iii) Multiply (2') by $-\frac{10}{13}$ and subtract from (3'), replace (3') with result, call it (3'')

$$\begin{bmatrix} 4 & 3 & 1 \\ 0 & -\frac{13}{4} & \frac{17}{4} \\ 0 & 5/2 & 3/2 \end{bmatrix} \to \begin{bmatrix} 4 & 3 & 1 \\ 0 & -\frac{13}{4} & \frac{17}{4} \\ 0 & 0 & \frac{248}{52} \end{bmatrix} = [U]$$

Recall that [L] is formed from the factor's that we multiplied row (1) + (2') by

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{2}{4} & -\frac{10}{13} & 1 \end{bmatrix}$$

② USE: $[L]\{d\}=\{b\}$ to solve for $\{d\}$ ②
using forward substitution:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{2}{4} & -\frac{10}{13} & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix}$$

$d_1 = 0$ from $d_1 = 0$

$d_2 = 4$ from $\frac{3}{4}d_1 + d_2 = 4$

$d_3 = \frac{53}{13}$ from $\frac{2}{4}d_1 - \frac{10}{13}d_2 + d_3 = 1$

STEP 3: Backward substitute using $[U]\{x\}=\{d\}$ to solve for $\{x\}$

$$\begin{bmatrix} 4 & 3 & 1 \\ 0 & -\frac{13}{4} & \frac{17}{4} \\ 0 & 0 & \frac{248}{52} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4 \\ 53/13 \end{Bmatrix}$$

$$\boxed{\begin{aligned} x_3 &= 0.8548 \\ x_2 &= -.1129 \\ x_1 &= -.1290 \end{aligned}}$$