Iterative Solvers
Gauss-Seidel

Ch. 11

Gauss-Seidel

- What is Gauss-Seidel?
  - Alternative to direct methods (like Gauss-Elim.)
  - Iterative approach
  - similar to the idea of successive substitution for root finding
- Why do we want to we want to use it?
  - Works well for large numbers of equations
  - Error is controlled by the number of iterations (Round off error is not usually as big a concern as Gauss-Elim)
  - Handles sparse matrices and large matrices well because it doesn't have to store all of the zero's.

Gauss-Seidel

- Approach, Consider

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

- Solve equation (1) for x_1, (2) for x_2 and (3) for x_3

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Gauss-Seidel - Method

- Start iteration process by guessing x^0_2 and x^0_3 and always using the most recent values of x's

$$x_1^1 \rightarrow x_2^0, x_2^0$$
$$x_2^1 \rightarrow x_1^1, x_3^0$$
$$x_3^1 \rightarrow x_1^1, x_2^1$$

Repeat using the new *x*'s

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$
$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$
$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

- Check for convergence:

$$\varepsilon_{a,i} = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} \right| \times 100\%$$

For all *i*'s where *k* = current iteration, *k-1* = previous iterations

Gauss-Seidel – Convergence criteria

- Diagonal Dominance – the diagonal element of a row should be greater than the sum of all other row elements

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

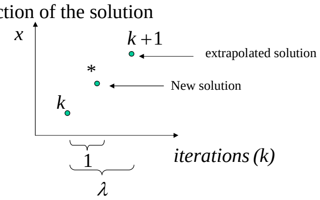
$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 7 \\ 3 & 1 & 1 \end{bmatrix}$$

Is this matrix diagonally dominant?

- Sufficient but not necessary (I.e., if the condition is satisfied, convergence is guaranteed, if the condition is NOT satisfied, convergence still may occur

Gauss-Seidel – Relaxation for iterative methods

- Motivation – speed up convergence – assuming we know the direction of the solution

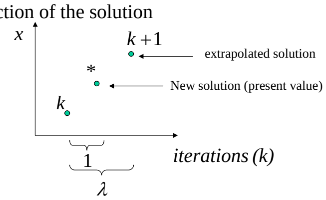


- Linear Extrapolation

$$\frac{x^* - x^k}{1} = \frac{x^{k+1} - x^k}{\lambda}$$
$$x^{k+1} = x^k + \lambda\left(x^* - x^k\right)$$
$$x^{k+1} = \lambda x^* + (1 - \lambda)x^k$$

Gauss-Seidel – Relaxation for iterative methods

- Motivation – speed up convergence – assuming we know the direction of the solution



- Linear Extrapolation

$$\frac{x^* - x^k}{1} = \frac{x^{k+1} - x^k}{\lambda}$$

$$x^{k+1} = x^k + \lambda\left(x^* - x^k\right) \qquad \lambda \text{ relaxation coefficient}$$

$$x^{k+1} = \lambda x^* + (1-\lambda)x^k$$

Gauss-Seidel – Relaxation for iterative methods

- What is x^* ?

$$x_1^* = \frac{b_1 - a_{12}x_2^k - a_{13}x_3^k}{a_{11}}$$

- Relaxation Coefficient – λ

$$x^{k+1} = \lambda x^* + (1-\lambda)x^k$$

  - Typical Values: $0 < \lambda < 2$
    1. No relaxation : $\lambda = 1$
    2. Underrelaxation: $0 < \lambda < 1$
       - Provides a weighted average of current & previous results
       - Used to make non-convergent systems converge
       - Helps speed up convergence by damping oscillations
    3. Overrelaxation: $1 < \lambda < 2$
       - Extra emphasis placed on present value
       - Assumes the solution is preceeding to the desired result, just too slowly
       - Will speed up convergence of a system that is already convergent
       - Selection of λ is problem specific and can require trial and error

Gauss-Seidel – Pseudocode

- 1st Loop: Row Normalization (divide all terms in the equation by diagonal term)
- 2nd Loop: Rearrange Equations:

$$x_1 = b_1 - a_{12}x_2 - a_{13}x_3 - \cdots$$

  - Note a11 = 1 after normalization
- Error checking flag – set to 1 at the beginning of each loop
  - Change to zero if any $\varepsilon_a > \varepsilon_s$
  - No need to check any ε_a's after that – saves computations

Show Matlab Gauss-Seidel Example

Other Iterative Solvers and GS varients

- Jacobi method – GS always uses the newest value of the variable x, Jacobi uses old values throughout the entire iteration
- Iterative Solvers are regularly used to solve Poisson's equation in 2 and 3D using finite difference/element/volume discretizations:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=f(x,y,z)$$

- Red Black Gauss Seidel
- Multigrid Methods

Engineering Application

- Determine the force in each member of the truss and the reaction forces