Boundary Value Problems

Ch. 27

Lecture Objectives

- To understand the difference between an initial value and boundary value ODE
- To be able to understand when and how to apply the shooting method and FD method.
- To understand what an Eigenvalue Problem is.

Initial Value Problems

- These are the types of problems we have been solving with RK methods

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2)$$

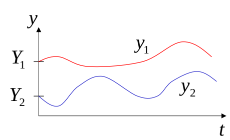
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2)$$

Subject to:

$t = 0$

$y_1(t=0) = Y_1$

$y_2(t=0) = Y_2$



Initial Value Problems

- These are the types of problems we have been solving with RK methods

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2)$$
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2)$$

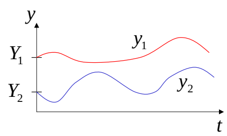
All conditions are specified at the same value of the independent variable!

Subject to:

$t = 0$

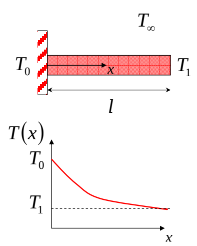
$y_1(t=0) = Y_1$

$y_2(t=0) = Y_2$



Boundary Value Problems

- Auxiliary conditions are specified at the boundaries (not just a one point like in initial value problems)

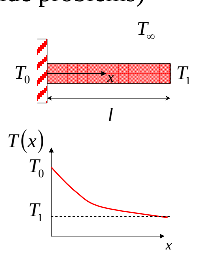


Two Methods:
Shooting Method
Finite Difference Method

Boundary Value Problems

- Auxiliary conditions are specified at the boundaries (not just a one point like in initial value problems)

conditions are specified at different values of the independent variable!

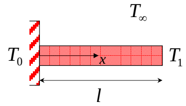


Two Methods:
Shooting Method
Finite Difference Method

Shooting Method

- Applicable to both linear & non-linear Boundary Value (BV) problems.
- Easy to implement
- No guarantee of convergence
- Approach:
  - Convert a BV problem into an initial value problem
  - Solve the resulting problem iteratively (trial & error)
  - Linear ODEs allow a quick linear interpolation
  - Non-linear ODEs will require an iterative approach similar to our root finding techniques.

Shooting Method – Cooling fin Example



h = heat transfer coefficient
k = thermal conductivity
P = perimeter of fin
A = cross sectional area of fin
T_∞ = ambient temperature

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0$$

$$T(x=0) = T_0$$
$$T(x=L) = T_1$$

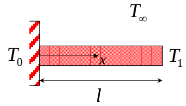
Analytical Solution

$$m^2 = \frac{hP}{kA}$$

$$\theta(x) = T(x) - T_\infty$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

Shooting Method – Cooling fin Example



$$\theta(x) = T(x) - T_\infty$$

$$m^2 = \frac{hP}{kA}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$\theta(x) = c_1 e^{mx} + c_2 e^{-mx}$$

Boundary Conditions

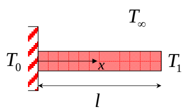
$$T(x=0) = T_0 \qquad \theta(x=0) = T_0 - T_\infty = \theta_0$$
$$T(x=L) = T_1 \qquad \theta(x=L) = T_1 - T_\infty = \theta_1$$

$$\theta(x=0) = c_1 + c_2 = \theta_0$$

$$\theta(x=L) = c_1 e^{mL} + c_2 e^{-mL} = \theta_1$$

$$\frac{\theta(x)}{\theta_0} = \frac{(\theta_1/\theta_0)\sinh mx + \sinh m(L-x)}{\sinh mL}$$

Shooting Method – Basic Method - Cooling fin Example



$$\frac{d^2T}{dx^2} - \frac{hP}{kA}\left(T - T_\infty\right) = 0$$

$$T(x=0) = T_0$$

$$T(x=L) = T_1$$

1. Rewrite as two first order ODEs

$$\frac{dT}{dx} = z$$

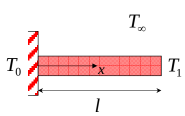
$$\frac{dz}{dx} = \frac{hP}{kA}\left(T - T_\infty\right)$$

2. We need an initial value for *z*, Guess:

$$T(x=0) = T_0$$

$$z(x=0) = z_1$$

Shooting Method – Basic Method - Cooling fin Example



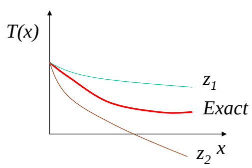
3. Integrate the two equations using RK4 and z_1 ; this will yield a solution at $x = l$

4. Integrate the two equations again using a 2nd guess guess for $z(x=0)=z_2$.

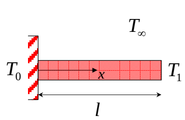
5. Linearly interpolate the *z* results to obtain the correct initial condition (Note: this only works for <u>Linear</u> ODEs.

Example:

$$z_{act} = z_2 + \frac{z_1 - z_2}{T_1 - T_2}\left(T_{act} - T_2\right)$$



Shooting Method – Cooling fin Example



$$T_\infty = 0$$

$$T_0 = 200$$

$$T_1 = 100$$

$$m^2 = \frac{hP}{kA} = 0.1$$

$$T(x) = 23.62e^{\sqrt{0.1}x} + 196.12e^{-\sqrt{0.1}x}$$

Matlab implementation (rung4_fin_multieqn.m)

Recast the problem:

$$T = y_1$$

$$\frac{dT}{dx} = y_2$$

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = m^2\left(y_1 - T_\infty\right)$$

$$y_1(x=0) = T_0 = 200$$

$$y_2(x=0) = G_1$$

Non-Linear BV Problems - Shooting Method

- Linear interpolation between 2 solutions will not necessarily result in a good estimate of the required boundary conditions
- Recast the problem as a Root finding problem
- The solution of a set of ODEs can be considered a function $g(z_o)$ where z_o is the initial condition that is unknown.

$$g(z_o) = f(z_o) - y_{bc}$$

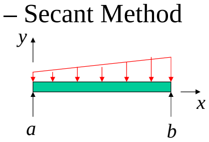
- Drive $g(z_o) \rightarrow 0$ to get our solution.
- Iteratively adjust your guess.

Non-linear Shooting method – Secant Method

Consider the following ODEs system

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = f(x, y, z) \qquad y(a) = 0 \qquad y(b) = y_b$$



1. Guess an initial value of z (i.e., $z(a)$) just as was done with the linear method. Using *RK4* or some other ODE method, we will obtain solution at $y(b)$.
2. Denote the difference between the boundary condition and our result from the integration as some function m.

$$m(z) = g(y(b), y'(b)) \longrightarrow$$ Find the zero of this function

$$m = y_{true}(b) - y_{guess}(b)$$

Non-linear Shooting method – Secant Method

3. Check to see if m is within an acceptable tolerance. Have we satisfied the boundary condition $y(b)$?

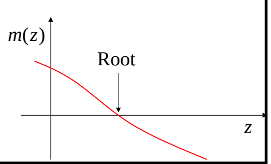
$$\varepsilon \le \varepsilon_s$$

$$\varepsilon = \left| \frac{m_i - m_{i-1}}{m_i} \right|$$

4. If not, then use the *Secant Method* to determine our next guess.

$$z_i = z_{i-1} - \frac{m(z_{i-1})}{m'(z_{i-1})}$$

$$z_i = z_{i-1} - \frac{z_{i-1} - z_{i-2}}{m_{i-1} - m_{i-2}} m_{i-1}$$



Non-linear Shooting method – Secant Method

$y(a) = 0$
$y(b) = y_b$ BV

↓

$z(a) = Guess$
$y(a) = 0$ IV

↓ Solve with RK4

$y(x)$
$z(x)$
$y(b)$

No ← $m = y_{true}(b) - y_{guess}(b)$ Is m small enough?

↓ Yes

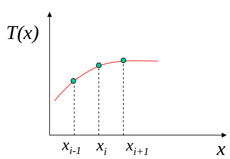
write out solution

Finite Difference Method

- Alternative to the shooting method
- Substitute finite difference equations for derivatives in the original ODE.
- This will give us a set of simultaneous algebraic equations that are solved a *nodes*.
- Recall using central differencing:

$$\frac{dT}{dx} = \frac{T_{i+1} - T_{i-1}}{2\Delta x}$$

$$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$



Finite Difference Method

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0$$

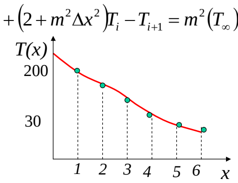
Rewrite in finite (central) difference form:

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - m^2(T_i - T_\infty) = 0$$

Multiply by Δx^2 and solve for T_i

$$T_{i+1} - 2T_i + T_{i-1} - m^2(T_i - T_\infty)\Delta x^2 = 0$$

$$-T_{i-1} + (2 + m^2\Delta x^2)T_i - T_{i+1} = m^2(T_\infty)\Delta x^2$$



$m^2 = 50$
$T_1 = 200$
$T_6 = 30$
$\Delta x = 0.1$

Finite Difference Method

General Equation:

$$-T_{i-1} + \left(2 + m^2 \Delta x^2\right) T_i - T_{i+1} = m^2 \left(T_\infty\right) \Delta x^2$$

Write out for all nodes:

$$-T_1 + \left(2 + m^2 \Delta x^2\right) T_2 - T_3 = m^2 \left(T_\infty\right) \Delta x^2$$
$$-T_2 + \left(2 + m^2 \Delta x^2\right) T_3 - T_4 = m^2 \left(T_\infty\right) \Delta x^2$$
$$-T_3 + \left(2 + m^2 \Delta x^2\right) T_4 - T_5 = m^2 \left(T_\infty\right) \Delta x^2$$
$$-T_4 + \left(2 + m^2 \Delta x^2\right) T_5 - T_6 = m^2 \left(T_\infty\right) \Delta x^2$$

Apply boundary conditions:

$m^2 = 50$
$T_1 = 200$
$T_5 = 30$
$\Delta x = 0.1$

$$\left.\begin{aligned} -200 + (2 + 0.5) T_2 - T_3 &= 12.5 \\ -T_2 + (2 + 0.5) T_3 - T_4 &= 12.5 \\ -T_3 + (2 + 0.5) T_4 - T_5 &= 12.5 \\ -T_4 + (2 + 0.5) T_5 - 30 &= 12.5 \end{aligned}\right\}$$

4 Equations, 4 unknowns

Finite Difference Method

Put in Matrix form:

$$\begin{bmatrix} 2.5 & -1 & 0 & 0 \\ -1 & 2.5 & -1 & 0 \\ 0 & -1 & 2.5 & -1 \\ 0 & 0 & -1 & 2.5 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 212.5 \\ 12.5 \\ 12.5 \\ 42.5 \end{Bmatrix}$$

Solve using one of our Systems of linear algebraic Equations methods

Fast, easy to implement technique for solving ODEs

Finite Difference Method – Extended to PDEs

Consider a simple Elliptical Equation: LaPlace's Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = R(x, y)$$

This could describe the steady state temperature distribution in 2D metal plate.

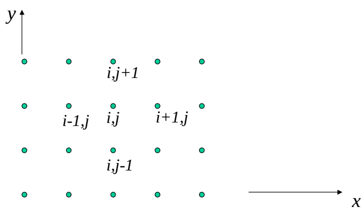
Discretize (write in finite difference form) our PDE using *Central Difference* technique:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_j + T_{j-1}}{\Delta y^2} = R_{i,j}$$

Finite Difference Method – Extended to PDEs

Consider a simple Elliptical Equation: LaPlace's Equation

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_j + T_{j-1}}{\Delta y^2} = R_{i,j}$$



Finite Difference Method – Extended to PDEs

Solve for $T_{i,j}$

$$T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + \frac{\Delta x^2}{\Delta y^2} T_{i,j+1} - 2T_{i,j} + T_{j-1} = \Delta x^2 R_{i,j}$$

If $\Delta x = \Delta y$, Uniform spacing

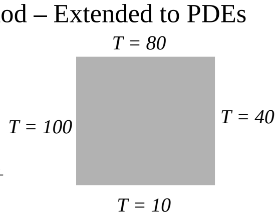
$$T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{j-1} = \Delta x^2 R_{i,j}$$

If $R = 0$

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

Finite Difference Method – Extended to PDEs

Suppose we have a heated plate with *Dirchlet boundary conditions*



$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

We can easily use Gauss-Seidel to solve our system of equations until:

$$\varepsilon \le \varepsilon_s$$

$$\varepsilon = \left| \frac{T_{i,j}^{new} - T_{i,j}^{old}}{T_{i,j}^{new}} \right|$$

Finite Difference Method – Extended to PDEs

Heated Plate Matlab Example