Roots of Polynomials

Ch. 7

Roots of Polynomials

General form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

n = order of the polynomial
a_i = constant coefficients

Roots – Real or Complex

1. For an n^{th} order polynomial – n real or complex roots
2. If n is odd → At least 1 real root
3. If complex roots exist, they are in complex conjugate pairs

$$\lambda + \mu i \qquad \lambda - \mu i$$

$$i = \sqrt{-1}$$

Roots of Polynomials

Polynomials

– Represent Mathematical models of real systems
– Result from characteristic equations of an ODE
  - The roots of the polynomial are Eigenvalues

Given a Homogeneous ODE:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad (1)$$

Solution is of the form:

$$y = e^{rt}$$

Roots of Polynomials

Polynomials

- Represent Mathematical models of real systems
- Result from characteristic equations of an ODE
  - The roots of the polynomial are Eigenvalues

Given a Homogeneous ODE (I.e. dynamic linear system):

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad (1)$$

Solution is of the form:

$$y = e^{rt} \longrightarrow$$ Substitute into equation (1)

Roots of Polynomials

$$a_2 r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} = 0$$

$$\boxed{a_2 r^2 + a_1 r + a_0 = 0}$$ Characteristic Equation

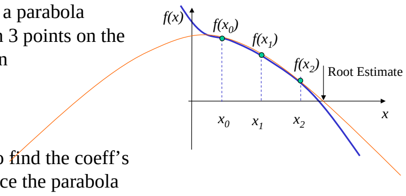
- The Roots → *r's* → Eigenvalues of the system
- Eigenvalues tell us important information about the system behavior.
- Roots represent important Engineering information
- For the Quadratic Case, the eigenvalues tells us:
  - *Overdamped* – 2 real roots
  - *Critically Damped* – one real root (the discriminant is zero)
  - *Underdamped* – 2 complex roots (the discriminant is negative)

Roots of Polynomials

- If only real Roots Exist:
  - Bracketing and Open Methods will work
  - Both methods require initial guesses
- If Roots are Complex
  - Bracketing methods will not work
  - Newton Raphson will work if the language handles complex numbers. (Still susceptible to diverging)
  - Introduce several New methods that avoid these problems.

Mueller's Method

- Project a parabola through 3 points on the function
- Need to find the coeff's that force the parabola through the 3 points
- Use the coefficients & quadratic formula to find where the parabola intersects the x-axis → Root Estimate



$$y = a_0 + a_1 x + a_2 x^2$$

Mueller's Method

- Given an Equation for a parabola

$$y = a_0 + a_1 x + a_2 x^2$$

- Rewrite relative to the point x_2

$$f(x) = c + b(x - x_2) + a(x - x_2)^2$$

- Substitute the 3 points (x_0, x_1, x_2) into the General equation

(1) $f(x_0) = c + b(x_0 - x_2) + a(x_0 - x_2)^2$

(2) $f(x_1) = c + b(x_1 - x_2) + a(x_1 - x_2)^2$

(3) $f(x_2) = c + b(x_2 - x_2) + a(x_2 - x_2)^2 = c$

3 Equations & Unknowns

Mueller's Method

- Substitute the value of c into (1) & (2)

(4) $f(x_0) - f(x_2) = b(x_0 - x_2) + a(x_0 - x_2)^2$

(5) $f(x_1) - f(x_2) = b(x_1 - x_2) + a(x_1 - x_2)^2$

- Solve the 2 Equations and 2 unknowns
- First, Make the following changes of variables to help us solve the equations

$$h_0 = x_1 - x_0 \qquad \delta_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$h_1 = x_2 - x_1 \qquad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Mueller's Method

- Next, add and subtract $f(x_1)$ to the LHS (4) and add and subtract x_1 to the RHS of (4)

$$[f(x_0)-f(x_1)]+[f(x_1)-f(x_2)]=b([x_0-x_1]+[x_1-x_2])+a([x_0-x_1]+[x_1-x_2])^2$$

$$f(x_1)-f(x_2)=b(x_1-x_2)+a(x_1-x_2)^2$$

$$h_0=x_1-x_0 \qquad \delta_0=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$h_1=x_2-x_1 \qquad \delta_1=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Mueller's Method

- Next, add and subtract $f(x_1)$ to the LHS (4) and add and subtract x_1 to the RHS of (4)

$$\underbrace{[f(x_0)-f(x_1)]}_{-\delta_0 h_0}+\underbrace{[f(x_1)-f(x_2)]}_{-\delta_1 h_1}=b(\underbrace{[x_0-x_1]}_{-h_0}+\underbrace{[x_1-x_2]}_{-h_1})+a(\underbrace{[x_0-x_1]}_{-h_0}+\underbrace{[x_1-x_2]}_{-h_1})^2$$

$$\underbrace{f(x_1)-f(x_2)}_{-\delta_1 h_1}=b\underbrace{(x_1-x_2)}_{-h_1}+a\underbrace{(x_1-x_2)}_{-h_1}^2$$

$$h_0=x_1-x_0 \qquad \delta_0=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$h_1=x_2-x_1 \qquad \delta_1=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Mueller's Method

Simplifying:

$$h_0\delta_0+h_1\delta_1=b(h_0+h_1)-a(h_0+h_1)^2$$

$$h_1\delta_1=h_1 b-ah_1^2$$

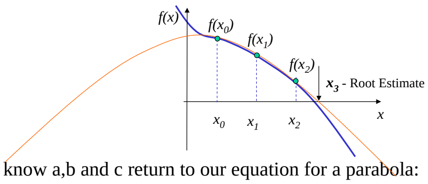
Solve for *a* and *b*:

$$a=\frac{\delta_1-\delta_0}{h_1+h_0}$$

$$b=ah_1+\delta_1$$

$$c=f(x_2)$$

Mueller's Method



Now that we know a,b and c return to our equation for a parabola:

$$f(x) = c + b(x - x_2) + a(x - x_2)^2 = 0$$

And solve for our root estimate x_3 using the Quadratic Formula

$$f(x_3) = c + b(x_3 - x_2) + a(x_3 - x_2)^2 = 0$$

Mueller's Method

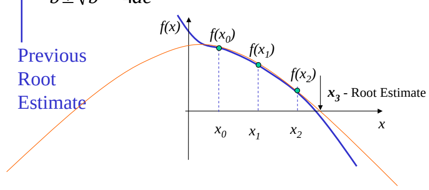
$$f(x_3) = c + b(x_3 - x_2) + a(x_3 - x_2)^2 = 0$$

Use alternative form of the Quadratic formula to reduce round-off error

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

Approximate Root (Current Estimate)

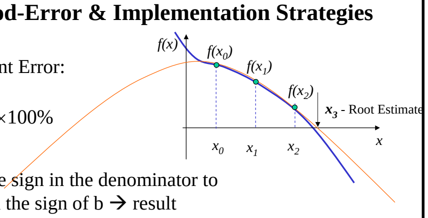
Previous Root Estimate



Mueller's Method-Error & Implementation Strategies

Relative Percent Error:

$$\varepsilon_a = \left| \frac{x_3 - x_2}{x_3} \right| \times 100\%$$



- Choose the sign in the denominator to agree with the sign of b → result largest denominator → x_3 will the root estimate closest to x_2
- 2 Strategies for discarding 1 of the x's when moving on to the next iteration
  1. Take the original 2 points closest to x_3
  2. Replace $x_0\, x_1\, x_2$ with $x_1\, x_2\, x_3$
     - Best when complex roots are needed

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

Mueller's Method Example

- Find the roots of

$$f(x) = x^3 - x^2 - 5x - 3$$

Mueller's Method - Notes

- Compared to Newton-Raphson, Mueller's method only requires function values, NOT derivatives.
- Will find complex roots
- Mueller's method can be used to find complex roots.
- Mueller's method fails when $f(x_1) = f(x_2) = f(x_3)$
- Rate of convergence is slightly less than quadratic.
- Can diverge

Review of Root Finding Methods

- <u>Bracketing Methods</u>
  - Graphical
  - Bisection
  - False Position
- 2 Guesses needed
- 2Always Converge
- Slow Convergence

- <u>Open Methods</u>
  - Successive Iteration
  - Newton Raphson
  - Modified Newton Raphson
  - Secant
- 1 Guess needed
- Possible Divergence
- Rapid Convergence

Review of Root Finding Methods

- Roots of Polynomials
  - Muller's Method (can handle imaginary roots)
- 3 Guesses needed
- Possible Divergence
- Rapid Convergence