Linear Algebraic Equations

Ch.9

Lecture Objectives

- To review the basic concepts of Matrix Mathematics
- To understand how to solve Engineering problems that involve the solution of systems of linear algebraic equations
- To understand what type of Engineering problems require systems of linear algebraic equations.
- To be able to implement your own systems of linear algebraic equations solver

Linear Algebraic Equations

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3 \\
&\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n
\end{aligned}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{Bmatrix}
$$

$$[A]\{x\} = \{b\}$$

Matrix Mathematics Review

Define:

- A_{ij} – elements of the array
- Rows – horizontal set of elements (*i's*)
- Columns – vertical set of elements (*j's*)
- Square Matrix – [*n x n*] on [*rows x columns*]
- Column vector

$$\{b\} \quad \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

- Symmetric Matrix: $a_{ij} = a_{ji}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 5 \end{bmatrix}$$

Matrix Mathematics Review

Matrix Dimensions: [*n x m*] on [*rows x columns*]

$$[5\times 2]: \begin{bmatrix} 6 & 1 \\ 3 & 1 \\ 2 & 4 \\ 9 & 11 \\ 0 & 3 \end{bmatrix} \qquad [4\times 1]: \begin{bmatrix} 3 \\ 2 \\ -4 \\ 7 \end{bmatrix} \qquad [1\times 6]: \begin{bmatrix} -8 & 1 & 7 & -5 & 0 & 2 \end{bmatrix}$$

Matrix Mathematics Review

Define:

7. Diagonal Matrix – Square matrix where off diagonal elements = 0.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

8. Identity Matrix - Square matrix where all elements along the main diagonal are 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Upper Triangular Matrix – All Elements below the main diagonal are zero.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Matrix Mathematics Review

Define:

10. Banded Matrix – All elements = 0 except in a band centered on the main diagonal
    - Bandwidth – 3,5 ,etc.
    - Tridiagonal → Gauss Seidel method

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

Matrix Mathematics Review

Define:

11. Addition

$$[A]+[B]=[C]$$

- A, B & C must have the same dimensions
- Element form: $c_{ij} = a_{ij} + b_{ij}$
- Commutative: $[A]+[B]=[B]+[A]$

12. Subtraction

$$[A]-[B]=[C]$$

- A, B & C must have the same dimensions
- Element form: $c_{ij} = a_{ij} - b_{ij}$
- Commutative: $[A]-[B]=-[B]+[A]$

Matrix Mathematics Review

Define:

13. Scalar Multiplication

$$g[A]=\begin{bmatrix} ga_{11} & ga_{12} & ga_{13} & \cdots & ga_{1n} \\ ga_{21} & ga_{22} & ga_{23} & \cdots & ga_{2n} \\ ga_{31} & ga_{32} & ga_{33} & \cdots & ga_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ga_{n1} & ga_{n2} & \cdots & \cdots & ga_{nn} \end{bmatrix}$$

Matrix Mathematics Review

Define:

14. Multiplication $[C]=[A][B]$

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$$

n is the column dimension of A and the row dimension of B

$$c_{11}=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\end{bmatrix}\begin{Bmatrix}b_{11}\\ b_{21}\\ b_{31}\\ b_{41}\end{Bmatrix}=a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+a_{13}\cdot b_{31}+a_{14}\cdot b_{41}$$

$$\begin{bmatrix}3 & 1 & 2 & 4\end{bmatrix}\begin{Bmatrix}1\\ 5\\ 3\\ 1\end{Bmatrix}=3\cdot 1+1\cdot 5+2\cdot 3+4\cdot 1=18$$

Matrix Mathematics Review

Define:

14. Multiplication

$$[A]_{n\times m}[B]_{m\times l}=[C]_{n\times l}$$

$$\begin{bmatrix}1 & 3 & 1\\ 2 & 1 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & -1\\ 3 & 1\end{bmatrix}=\begin{bmatrix}1\cdot 2+3\cdot 3+1\cdot 3 & 1\cdot 1+3\cdot -1+1\cdot 1\\ 2\cdot 2+1\cdot 3+4\cdot 3 & 2\cdot 1+1\cdot -1+4\cdot 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & 1\\ 2 & 1 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & -1\\ 3 & 1\end{bmatrix}=\begin{bmatrix}14 & -1\\ 19 & 5\end{bmatrix}$$

Matrix Mathematics Review

Define:

14. Multiplication

- Associative

  $([A][B])[C]=[A]([B][C])$

- Distributive

  $[A]([B]+[C])=[A][B]+[A][C]$

- NOT Commutative – Order does Matter

  $[A][B]\neq[B][A]$

Matlab error → inner matrix dimensions must agree

Matrix Mathematics Review

Define:

15. Division – Use Matrix Inversion
    - For a Square non-singular matrix

$$[A][A]^{-1} = [I]$$

$$[A]^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- For a System of Equations

$$[A]\{x\} = \{b\}$$

$$\{x\} = [A]^{-1}\{b\}$$

Matrix Mathematics Review

Define:

16. <u>Transpose</u>: transforming rows into columns

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

Matrix Mathematics Review

Define:

16. <u>Transpose</u>: transforming rows into columns

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$[A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

Matrix Mathematics Review

Define:

17. Trace: sum of the elements along the diagonal

$$tr[A] = \sum_{i=1}^{n} a_{ii}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Gauss Elimination

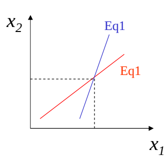
- Combining Equations to eliminate unkowns
- Introduce 3 methods that are useful for small numbers of equations ($n \le 3$)

1. **Graphical Method**
2. **Cramer's Rule**
3. **Elimination of Unknowns**

Gauss Elimination - Graphical

- Best for 2 equations

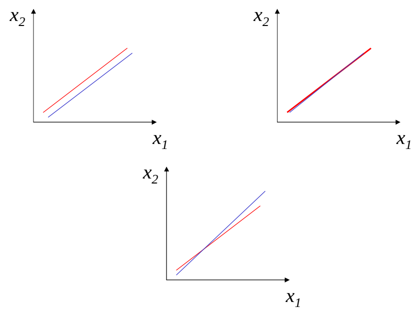
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$



Point of intersection represents the solution

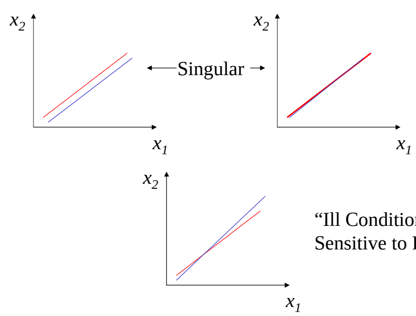
Gauss Elimination - Graphical

- Problems



Gauss Elimination - Graphical

- Problems



“Ill Conditioned”
Sensitive to RO error

Gauss Elimination – Cramer’s Rule

Gauss Elimination – Elimination of Unknowns

$n \leq 3$