Naïve Gauss Elimination

Ch.9

Naïve Gauss Elimination
Linear Algebra Review

Elementary Matrix Operations Needed for Elimination Methods:

- Multiply an equation in the system by a non-zero real number.
- Interchange the positions of two equation in the system.
- Replace an equation by the sum of itself and a multiple of another equation of the system.

Naïve Gauss Elimination
Similar to Elimination of Unknowns

1. Forward Elimination
2. Backward Substitution

Naïve because we don't consider division by zero to be a possibility

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Naïve Gauss Elimination
Similar to Elimination of Unknowns

1. Forward Elimination of Unknowns
   1. Reduce the coeficient matrix [A] to an upper triangular system
   2. Eliminate x_1 from the 2nd to nth Eqns.
   3. Eliminate x_2 from the 3rd to nth Eqns.
   4. Continue process until the nth equation has only 1 Non-Zero coefficient

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

Naïve Gauss Elimination

1. Forward Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \qquad (1)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \qquad (2)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \qquad (3)$$

Eliminate x_1 from equation (2). Multiply (1) by a_{21}/a_{11}, then subtract the result from (2)

$$-\left(\frac{a_{21}}{a_{11}}a_{11}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \frac{a_{21}}{a_{11}}a_{13}x_3 = \frac{a_{21}}{a_{11}}b_1\right)$$
$$a_{21}x_1 + \quad a_{22}x_2 + \quad a_{23}x_3 = b_2$$
$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)x_3 = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$$a'_{22}\,x_2 + \quad a'_{23}\,x_3 = b'_2 \qquad (2')$$

Naïve Gauss Elimination

1. Forward Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \qquad (1)$$ ← Pivot Equation
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \qquad (2)$$ ← Elimination Row
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \qquad (3)$$

Eliminate x_1 from equation (2). Multiply (1) by a_{21}/a_{11}, then subtract the result from (2)

$$-\left(\frac{a_{21}}{a_{11}}a_{11}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \frac{a_{21}}{a_{11}}a_{13}x_3 = \frac{a_{21}}{a_{11}}b_1\right)$$

Pivot Element (← a_{11})

$$a_{21}x_1 + \quad a_{22}x_2 + \quad a_{23}x_3 = b_2$$
$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13}\right)x_3 = b_2 - \frac{a_{21}}{a_{11}}b_1$$

$$a'_{22}\,x_2 + \quad a'_{23}\,x_3 = b'_2 \qquad (2')$$

Naïve Gauss Elimination

1. Forward Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$
$$+a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad (2')$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad (3) \leftarrow \text{Elimination Row}$$

Eliminate x_1 from (3). Multiply (1) by a_{31}/a_{11}, then subtract the result from (3)

$$-\left(\frac{a_{31}}{a_{11}}a_{11}x_1 + \frac{a_{31}}{a_{11}}a_{12}x_2 + \frac{a_{31}}{a_{11}}a_{13}x_3 = \frac{a_{31}}{a_{11}}b_1\right)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$
$$\left(a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right)x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}}a_{13}\right)x_3 = b_3 - \frac{a_{31}}{a_{11}}b_1$$

$$\boxed{a'_{32}x_2 + a'_{33}x_3 = b'_3}$$

Naïve Gauss Elimination

1. Forward Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$
$$+a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad (2') \leftarrow \text{Pivot Equation}$$
$$+a'_{32}x_2 + a'_{33}x_3 = b'_3 \quad (3') \leftarrow \text{Elimination Row}$$

Eliminate x_2 from (3'). Multiply (2') by a'_{32}/a'_{22}, then subtract the result from (3')

$$-\left(\frac{a'_{32}}{a'_{22}}a'_{22}x_2 + \frac{a'_{32}}{a'_{22}}a'_{23}x_3 = \frac{a'_{32}}{a'_{22}}b'_2\right)$$
$$a'_{32}x_2 + a'_{33}x_3 = b'_3$$
$$+\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 = b'_3 - \frac{a'_{32}}{a'_{22}}b'_2$$

$$\boxed{a''_{33}x_3 = b''_3}$$

Naïve Gauss Elimination

1. Forward Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (1)$$
$$a'_{22}x_2 + a'_{23}x_3 = b'_2 \quad (2')$$
$$+a''_{33}x_3 = b''_3 \quad (3'')$$

Solve for x_3

$$x_3 = \frac{b''_3}{a''_{33}}$$

Naïve Gauss Elimination

2. Backwards substitution:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \qquad (1)$$
$$a'_{22}\,x_2 + a'_{23}\,x_3 = b'_2 \qquad (2')$$
$$+ a''_{33}\,x_3 = b''_3 \qquad (3'')$$

$$x_3 = \frac{b''_3}{a''_{33}}$$

From (2')

$$x_2 = \frac{b'_2 - a'_{23}\,x_2}{a'_{22}}$$

From (1)

$$x_3 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

Naïve Gauss Elimination

In General, the last equation should reduce to:

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

General form is how we will numerically implement.

Note: Since we normalize with the pivot element, if it is zero, we have a problem → Naïve method

Naïve Gauss Elimination

Example:

$$2x_1 + x_2 + 3x_3 = 1 \qquad (1)$$
$$4x_1 + 4x_2 + 7x_3 = 1 \qquad (2)$$
$$2x_1 + 5x_2 + 9x_3 = 3 \qquad (3)$$

Naïve Gauss Elimination – Numerically Implementing

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

3 Main Loops: Forward Elimination

1. Pivot Row – from 1st row to the n-1 row, move down, we will call the pivot row, row k.
2. Elimination Row – Rows below Pivot row, where eliminations take place (top down), call this the ith row.
3. Element transform Loop – columns, jth column. Move left to right.

Naïve Gauss Elimination – Numerically Implementing

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

What we do: Forward Elimination

A. Normalization Step: multiply the kth row elements a_{kj} by $(-a_{ik}/a_{kk})$
B. Add the result of step A, to a_{ij}

$$a'_{ij} = a_{ij} - \frac{a_{ik}}{a_{kk}} a_{kj}$$ ← Instead of saving a'_{ij} we save as a_{ij}

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

C. Calculate the new b's or right hand side terms $b'_i = b_i - b_k \dfrac{a_{ik}}{a_{kk}}$

$$b'_2 = b_2 - b_1 \frac{a_{21}}{a_{11}}$$

Naïve Gauss Elimination – Numerically Implementing

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

What we do: Back Substitution

For the nth Row:

$$x_n = \frac{b_n}{a_{nn}}$$

Now, work backwards row by row, right to left (n-1 row and n column)

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

Naïve Gauss Elimination – Pseudocode

```
%Forward Elimination to build an upper triangular matrix
for k=1:n-1
   for i=k+1:n
      factor = a(i,k)/a(k,k);                 %normalzing factor (Step A)
      for j=k+1:n                             %move accross the columns loop
         a(i,j) = a(i,j) - factor*a(k,j);     %(Step B)
      end
      b(i)=b(i)-factor*b(k);                  %(Step C)
   end
end
%Backward Substitution
x(n)=b(n)/a(n,n);                             %solve for the last x value
for i=n-1:-1:1
   sum = 0;
   for j=i+1:n
      sum = sum + a(i,j)*x(j);
   end
   x(i)=(b(i)-sum)/a(i,i);
end
```

Problems with Naïve Elimination Methods

1. Division by zero
   - if a_{11} = 0, then the 1st elimination step yields division by zero.
   - "pivoting" technique will be used to avoid this problem
2. R.O. Error
   - Every result is dependant on previous results →RO error can propagate
   - "Rule of Thumb" – if n > 100
   - Double precision will help
3. Ill Conditioned Systems (D~0)
   - Small changes in the coefficient (a_{ij}) matrix result in large changes in the solution
   - Or, alternatively a wide range of answers (x_i's) satisfy the equations
   - RO error can produce small changes in coefficients that can lead to large errors, (Check by slightly changing the coefficients and seeing the effect on the results)

Problems with Naïve Elimination Methods

4. Singular Systems: (D=0)
   - One or more equations are identical
   - We have (*n-1*) equations and *n* unknowns

QUICK way to check D

- After the forward elimination evaluate the determinant of the modified coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{vmatrix} = a_{11} \cdot a'_{22} \cdot a''_{33}$$

Problems with Naïve Elimination Methods

When checking D, how small is too small? Solution: Standardize the determinant.

Scale Equations – such that the maximum coefficient for any equation is 1.

$$\begin{aligned} 3x_1 + 2x_2 &= 12 \\ -x_1 + 3x_2 &= 3 \end{aligned} \longrightarrow \quad D = 11$$

Divide (1) by 3 and (2) by 3.

$$\begin{aligned} x_1 + \frac{2}{3}x_2 &= 4 \\ -\frac{1}{3}x_1 + x_2 &= 1 \end{aligned} \longrightarrow \quad D = 1.22$$

Problems with Naïve Elimination Methods

Now for a 2x2 matrix D takes on Values:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= \mathrm{b}_1 \\ a_{21}x_1 + a_{22}x_2 &= \mathrm{b}_2 \end{aligned}$$

$$D = a_{11}a_{22} - a_{12}a_{21}$$

$$|D| \le 2$$

Methods for Improving Solutions

1. Use More Significant digits
2. Partial Pivoting

- Avoid division by zero or vary small numbers
  a) Before normalizing in Gauss elimination, find the largest element (absolute value)in the first column
  b) Reorder the equations so that the largest element is the pivot element
  c) Repeat for each elimination step – I.e., 2^{nd} application would find the largest element in the 2^{nd} column (below the 1^{st} Equation) and seek the largest pivot element.

Methods for Improving Solutions – Partial Pivoting code

```
p=k;                          %assume row with largest coefficient
big=abs(a(k,k))               %assume the diagnol term is largest

for ii = k+1:n                %move down the rows to check elements
   dummy=abs(a(ii,k));
   if dummy > big             %if the elent is bigger swap it out
      big=dummy;
      p=ii;                   %rename the largest row
   end
end                           %end for loop

%if p is not equal to k, we need to swap row k with row p
%if p is equal to k, then we don't do anything
if p~=k
   for jj=k:n                 %move across columns to swap coefficient values
      dummy=a(p,jj);          %temporarily store the element
      a(p,jj)=a(k,jj);
      a(k,jj)=dummy;
   end
   dummy = b(p);              %now sway right hand side values
   b(p) = b(k);
   b(k) = dummy;
end
```

Show Matlab example

Methods for Improving Solutions

3. Scaling – helps make pivoting decisions

$$3x_1 + 70{,}000x_2 = 40{,}000$$
$$-x_1 + .2x_2 = 3$$

What problem could Arise here

↓ scale

$$0.0000428x_1 + 1x_2 = 0.5714$$
$$-x_1 + .2x_2 = 3$$

↓ Pivot

$$-x_1 + .2x_2 = 3$$
$$0.0000428x_1 + 1x_2 = 0.5714$$

Methods for Improving Solutions

3. Scaling –
   - We have seen that adding and subtracting of numbers with very different magnitudes can result in RO error
   - Scale all rows so that the maximum coefficient value in any row is one.
   - NOTE: scaling by very large numbers can potentially introduce RO error

Suggestion:
- Employ scaling only to make a decision regarding pivoting
- Comparison & row switching are not subject to RO error
- Complete solution using original coefficients

Gauss-Jordan Elimination

- Variation of Gauss Elimination
- When an unknown is eliminated it is eliminated from all equations, not just subsequent ones (Diagonal Matrix Results)
- All rows are normalized by their pivot element
- Identity Matrix results

$[A]$ Is augmented $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

$$[I]\{x\} = \{b'\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{Bmatrix}$$

- Almost identical to Gauss Elimination but, more operations are required
- No back substitution step

Methods for Improving Solutions – Partial Pivoting Overhead Gauss Jordan example