Differential Forms of the Conservation Laws
Momentum

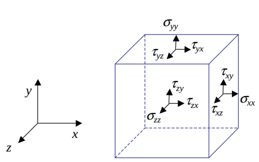
Differential Forms of the Conservation Laws

- <u>Field Approach</u> - Powerful tool that allows us to find detailed information about flow fields. Eularian viewpoint
- Solutions to the differential forms of the conservation laws will yield the spatial distribution of various important quantities in fluid mechanics. i.e., V(x,y,z,t), P(x,y,z,t), T(x,y,z,t)

Cauchy's Equation of Motion
Stresses on a Fluid Element

<u>Stress Tensor</u>
•Symmetric
•Normal stresses along the diagonal
•Includes surface & body forces acting on a fluid element

$$\dot{T}=\begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$



Apply Newton's 2nd Law for a differential fluid element

Cartesian Coordinate System

Cauchy's Equation of Motion

x - direction :

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x+\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$

y - direction :

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y+\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)$$

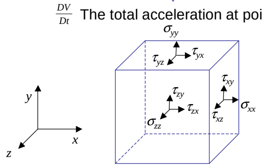
z - direction :

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z+\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right)$$

$\frac{DV}{Dt}$ The total acceleration at point

Body force per Volume

Stress gradients or stress divergence



Good but, if we include conservation of mass we still have 9 unknowns and only 4 equations!

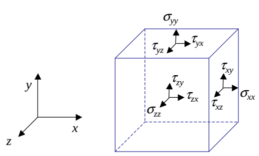
Cartesian Coordinate System

Kinematics of Fluid Motion - A way to relate stresses to velocities and reduce our number of unknowns

- Translation
- Rotation
- Deformation
- Dilatation

True no matter what forces cause the motion

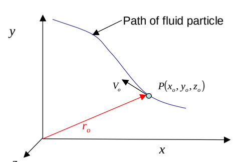
Historically:
Cauchy (1841) – from solid body deformation analysis
Stokes (1845) – viscous fluids & the relationship between stress & strain
Helmoltz (1858) – analyzed the deformation of ideal fluids - vorticity



Multimedia CD – search word *fluid particle*

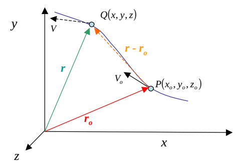
Velocity Composition

Consider a particle of fluid moving along a path.



Velocity Composition

Taylor Series Expansion of the velocity vector about the point P.



u - velocity component:

$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o + HOT$$

v - velocity component:

$$v = v_o + (x - x_o)\left(\frac{\partial v}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial v}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial v}{\partial z}\right)_o + HOT$$

w - velocity component:

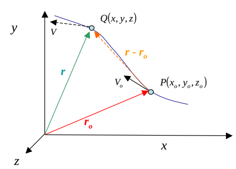
$$w = w_o + (x - x_o)\left(\frac{\partial w}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial w}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial w}{\partial z}\right)_o + HOT$$

If $(r - r_o)$ is small, we can neglect the HOT

Velocity Composition

Now a bit of Mathematical Jugglery:

1. Break up the partial differential terms

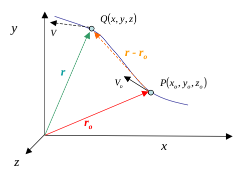


$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o$$

Velocity Composition

Now a bit of Mathematical Jugglery:

1. Break up the partial differential terms



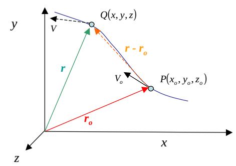
$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o$$

$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right]$$

Velocity Composition

Now a bit of Mathematical Jugglery:

1. Break up the partial differential terms
2. Add and subtract like differentials



$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o$$

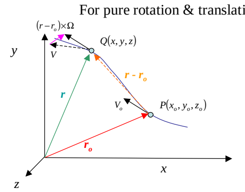
$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right]$$

$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_o\right]$$

Velocity Composition

We now have a new expression for the velocity vector V

For pure rotation & translation



$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)_o\right]$$

$$V = V_o + (r - r_o)\times\dot{\Omega} + (r - r_o)\times\dot{S}$$

Rotation Rate

Strain Rate – related to the deformation and dilatation of fluid elements

Velocity Composition

$$V = V_o + (r - r_o)\times\dot{\Omega} + (r - r_o)\times\dot{S}$$

Rotation Rate

Strain Rate

$$\dot{\Omega} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}$$

Asymmetric

$$\dot{S} = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \left(\frac{\partial v}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \left(\frac{\partial w}{\partial z}\right) \end{bmatrix}$$

Symmetric

Since the stress tensor T is also symmetric, T can only be related to the Strain Rate

Relating Velocity Changes & Stress

The Constitutive Equation of Fluid Mechanics

$$\dot{T} = a\dot{S} + b\dot{I}$$

Similar to Hooke's Law in solid mechanics, for a Newtonian Fluid

$$\dot{T} = 2\mu\dot{S} - \left(p + \frac{2}{3}\mu(\nabla \bullet V)\right)\dot{I}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \sigma_{xx} = -p - \frac{2}{3}\mu(\nabla \bullet V) + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \sigma_{yy} = -p - \frac{2}{3}\mu(\nabla \bullet V) + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \sigma_{zz} = -p - \frac{2}{3}\mu(\nabla \bullet V) + 2\mu\frac{\partial w}{\partial z}$$

Navier-Stokes Equations
(for an incompressible fluid & constant viscosity)

x - direction :

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y - direction :

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z - direction :

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

4 Equations & 4 unknowns → u,v,w and P

We now have a closed system of Equations!!!

Roadmap to Navier-Stokes Equations

1. Conservation of mass
2. Conservation of momentum
   a. Total acceleration of fluid particle
   b. Forces on Fluid Element: Surface & Body Gravitational force
   c. Yields Cauchy's equation
   d. Kinematics gives us a relationship between stress and strain.