

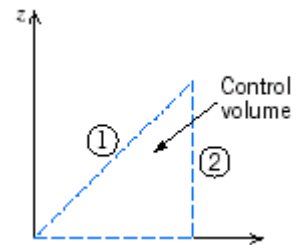
Problem 4.10

The velocity field in the region shown is given by $\vec{V} = az\hat{j} + b\hat{k}$, where $a = 10 \text{ s}^{-1}$ and $b = 5 \text{ m/s}$. For the $1 \text{ m} \times 1 \text{ m}$ triangular control volume (depth $w = 1 \text{ m}$ perpendicular to the diagram), an element of area ① may be represented by $w(-dz\hat{j} + dy\hat{k})$ and an element of area ② by $w dz\hat{j}$.

- (a) Find an expression for $\vec{V} \cdot d\vec{A}_1$.
- (b) Evaluate $\int_{A_1} \vec{V} \cdot d\vec{A}_1$.
- (c) Find an expression for $\vec{V} \cdot d\vec{A}_2$.
- (d) Find an expression for $\vec{V}(\vec{V} \cdot d\vec{A}_2)$.
- (e) Evaluate $\int_{A_2} \vec{V}(\vec{V} \cdot d\vec{A}_2)$.

Given: Data on velocity field and control volume geometry

Find: Several surface integrals



Solution

$$\begin{array}{ll} dA_1 | 4 dz\hat{j} + 2 dy\hat{k} & dA_1 | 4 dz\hat{j} + 2 dy\hat{k} \\ dA_2 | w dz\hat{j} & dA_2 | dz\hat{j} \\ \vec{V} | az\hat{j} + b\hat{k} & \vec{V} | 10z\hat{j} + 5\hat{k} \end{array}$$

$$(a) \int_{A_1} \vec{V} \cdot dA_1 = \int_0^1 \int_0^1 (10z\hat{j} + 5\hat{k}) \cdot (4 dz\hat{j} + 2 dy\hat{k}) = \int_0^1 \int_0^1 (40z + 10) dz dy = 40 \int_0^1 \left[\frac{z^2}{2} + z \right]_0^1 dy = 40 \int_0^1 \left(\frac{1}{2} + 1 \right) dy = 40 \int_0^1 \frac{3}{2} dy = 60 \int_0^1 dy = 60 \text{ m}^2/\text{s}$$

$$(b) \int_{A_1} \vec{V} \cdot dA_1 = 4 \int_0^1 \int_0^1 (10z + 5) dz dy = 4 \left[5z^2 + 5z \right]_0^1 \Big|_0^1 = 4 (5 + 5) = 40 \text{ m}^2/\text{s}$$

$$(c) \int_{A_2} \vec{V} \cdot dA_2 = \int_0^1 (10z\hat{j} + 5\hat{k}) \cdot dz\hat{j} = \int_0^1 10z dz = 5z^2 \Big|_0^1 = 5 \text{ m}^2/\text{s}$$

$$(d) \int_{A_2} \vec{V}(\vec{V} \cdot dA_2) = \int_0^1 (10z\hat{j} + 5\hat{k})(10z) dz = \int_0^1 (100z^2\hat{j} + 50z\hat{k}) dz = \left[\frac{100}{3}z^3\hat{j} + 25z^2\hat{k} \right]_0^1 = \frac{100}{3}\hat{j} + 25\hat{k}$$

$$(e) \int_{A_2} \vec{V}(\vec{V} \cdot dA_2) = \int_0^1 (10z\hat{j} + 5\hat{k})(10z) dz = \left[\frac{100}{3}z^3\hat{j} + 25z^2\hat{k} \right]_0^1 = 33.3\hat{j} + 25\hat{k}$$