Problem 6.3

Given: Horizontal flow of water described by the velocity field

$$\vec{V} = (Ax + Bt)\hat{i} + (-Ay + Bt)\hat{j}$$

where: $A = 5\,s^{-1}$, $B = 10\,ft\cdot s^{-2}$, coordinates x, y in ft, t in s.

Find: (a) Expressions for (i) local, (ii) convective, (iii) total, acceleration
(b) Evaluate at point (2,2) for $t = 5\,s$
(c) Evaluate ∇p at same point and time

Solution:

Basic equations: $\dfrac{D\vec{V}}{Dt} = \vec{a}_p = \underbrace{\dfrac{\partial \vec{V}}{\partial t}}_{\text{local}} + \underbrace{u\dfrac{\partial \vec{V}}{\partial x} + v\dfrac{\partial \vec{V}}{\partial y} + \cancel{w\dfrac{\partial \vec{V}}{\partial z}}^{\,0}}_{\text{convective}}$; $\rho\vec{g} - \nabla p = \rho\dfrac{D\vec{V}}{Dt}$

Assumptions: (1) frictionless flow
(2) ρ = constant = 1.94 slug/ft^3

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right] = B\hat{i} + B\hat{j} = 10(\hat{i}+\hat{j})\ ft/s^2 \leftarrow \vec{a}_{local}$$

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = (Ax+Bt)\frac{\partial}{\partial x}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right] + (-Ay+Bt)\frac{\partial}{\partial y}\left[(Ax+Bt)\hat{i} + (-Ay+Bt)\hat{j}\right]$$

$$= (Ax+Bt)\left[A\hat{i}\right] + (-Ay+Bt)\left[-A\hat{j}\right]$$

$$u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} = A(Ax+Bt)\hat{i} - A(-Ay+Bt)\hat{j} \leftarrow$$

$$= \frac{5}{s}\left(\frac{5}{s}\times 2\,ft + \frac{10\,ft}{s^2}\times 5\,s\right)\hat{i} - \frac{5}{s}\left(-\frac{5}{s}\times 2\,ft + \frac{10\,ft}{s^2}\times 5\,s\right)\hat{j} = 300\hat{i} - 200\hat{j}\ \frac{ft}{s^2} \leftarrow \vec{a}_{conv}$$

$$\vec{a} = \vec{a}_{local} + \vec{a}_{conv} = \left[B + A(Ax+Bt)\right]\hat{i} + \left[B - A(-Ay+Bt)\right]\hat{j} = 310\hat{i} - 190\hat{j}\ \frac{ft}{s^2} \leftarrow \vec{a}$$

From Euler's equation,

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt} = 1.94\,\frac{slug}{ft^3}\left[-32.2\hat{k} - (310\hat{i} - 190\hat{j})\right]\frac{ft}{s^2}\times\frac{lbf\cdot s^2}{ft\cdot slug}$$

$$\nabla p = -601\hat{i} + 367\hat{j} - 62\hat{k}\ \frac{lbf/ft^2}{ft} = -4.17\hat{i} + 2.56\hat{j} - 0.43\hat{k}\ psi/ft$$

Note: $\nabla\cdot\vec{V} = 0$ as required for incompressible flow