Problem 9.10

Given: Linear, parabolic, and sinusoidal profiles used to represent the laminar boundary layer velocity profile.

Evaluate: the ratio θ/δ for each profile.

Solution:

Definition: $\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$ (9.2)

then,

$$\theta/\delta = \frac{1}{\delta}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta$$

Linear profile $\frac{u}{U} = y/\delta = \eta$

$$\theta/\delta = \int_0^1 \eta(1-\eta)\,d\eta = \int_0^1 (\eta-\eta^2)\,d\eta = \left[\frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right]_0^1 = \frac{1}{6} = 0.167$$

Parabolic profile $\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$

$$\theta/\delta = \int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta = \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,d\eta$$

$$\theta/\delta = \left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1 = \left[1 - \frac{5}{3} + 1 - \frac{1}{5}\right] = \frac{2}{15} = 0.133$$

Sinusoidal profile $\frac{u}{U} = \sin\frac{\pi}{2}\frac{y}{\delta} = \sin\frac{\pi}{2}\eta$

$$\theta/\delta = \int_0^1 \sin\frac{\pi}{2}\eta\left(1-\sin\frac{\pi}{2}\eta\right)d\eta = \int_0^1 \left(\sin\frac{\pi}{2}\eta - \sin^2\frac{\pi}{2}\eta\right)d\eta$$

$$\theta/\delta = \left[-\frac{2}{\pi}\cos\frac{\pi}{2}\eta - \frac{2}{\pi}\left\{\frac{\pi\eta}{4} - \frac{1}{4}\sin\pi\eta\right\}\right]_0^1 = -0-\left(-\frac{2}{\pi}\right) - \frac{2}{\pi}\left(\frac{\pi}{4}\right) - 0$$

$$\theta/\delta = \frac{2}{\pi} - \frac{1}{2} = 0.137$$

Summarizing:

Profile	Expression	θ/δ
Linear	$u/U = \eta$	0.167
Parabolic	$u/U = 2\eta - \eta^2$	0.133
Sinusoidal	$u/U = \sin\frac{\pi}{2}\eta$	0.137