Derivation of the Differential Forms of the Conservation Laws
Momentum

Approach to Deriving the Differential Forms of the Conservation Laws

1. Write out the law for a system of particles

$$\frac{DN_{sys}}{Dt} = \frac{D}{Dt}\int_V \rho\eta d\forall = \cdots$$

2. Rewrite the law in terms of a control volume using the R.T.T. and Leibniz's Theorem

$$\frac{DN_{sys}}{Dt} = \int_{CV} \frac{\partial(\eta\rho)}{\partial t} d\forall + \int_{CS} \eta\rho(\vec{V}\cdot\hat{n})dA$$

3. Use Gauss's Theorem to transform area integrals into volume integrals, so that the law may be written in the form:

$$\int_{CV} \{\cdots\} d\forall = 0$$

Deriving the Differential Forms of the Conservation Laws

4. For an arbitrary control volume the integrand must be zero yielding the differential equation:

$$\{\cdots\} = 0$$

5. Simplify and Recast equations into the most usable form

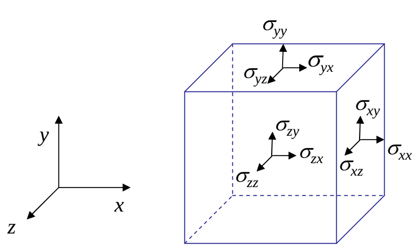
Cauchy's Equation of Motion

Stresses on a Fluid Element

Stress Tensor
•Symmetric
•Normal stresses along the diagonal
•Includes surface & body forces acting on a fluid element

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

Apply Newton's 2nd Law for a differential fluid element



Cartesian Coordinate System

Stresses on a Fluid Element
Proof of Symmetry

Consider a Torque on the element below about an axis along the centroid parallel to the z-axis

$$Torque = \left[\sigma_{xy} + \frac{1}{2}\frac{\partial \sigma_{xy}}{\partial x}dx\right][dydz]\frac{dx}{2} + \left[\sigma_{xy} - \frac{1}{2}\frac{\partial \sigma_{xy}}{\partial x}dx\right][dydz]\frac{dx}{2}$$

$$-\left[\sigma_{yx} + \frac{1}{2}\frac{\partial \sigma_{yx}}{\partial x}dy\right][dxdz]\frac{dy}{2} - \left[\sigma_{yx} - \frac{1}{2}\frac{\partial \sigma_{yx}}{\partial x}dy\right][dxdz]\frac{dy}{2}$$

$$Torque = (\sigma_{xy} - \sigma_{yx})dxdydz = I\alpha_z$$

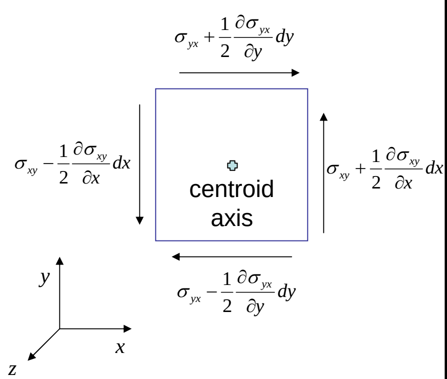
Moment of Inertia

$$I = dxdydz(dx^2 + dy^2)\rho/12$$

$$(\sigma_{xy} - \sigma_{yx}) = \frac{\rho}{12}(dx^2 + dy^2)\alpha_z$$

As dx, dy go to zero

$$\sigma_{xy} = \sigma_{yx}$$



Cauchy's Equation of Motion

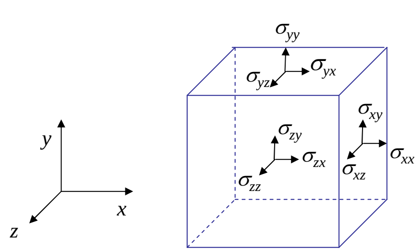
$$\underbrace{\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_j u_k}{\partial x_k}}_{} = \underbrace{\frac{\partial \sigma_{ij}}{\partial x_i}}_{} + \underbrace{\rho f_j}_{}$$

$\frac{D\rho\vec{V}}{Dt}$ The total acceleration at point

Stress gradients or stress divergence

Body force per Volume

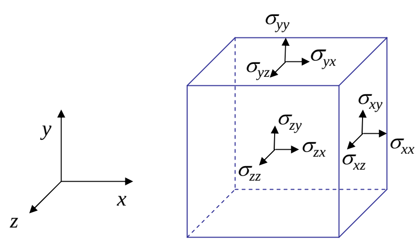
Good but, if we include conservation of mass we still have 9 unknowns and only 4 equations!



Cartesian Coordinate System

Kinematics of Fluid Motion - A way to relate stresses to velocities and reduce our number of unknowns

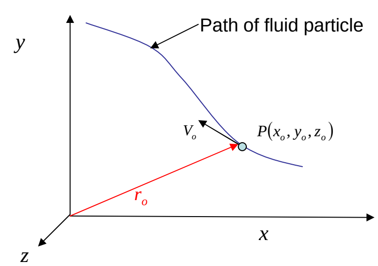
- Translation
- Rotation
- Deformation
- Dilatation



Historically:
Cauchy (1841) – from solid body deformation analysis
Stokes (1845) – viscous fluids & the relationship between stress & strain
Helmoltz (1858) – analyzed the deformation of ideal fluids - vorticity

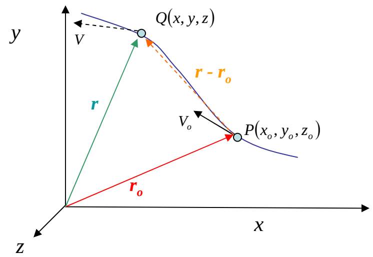
Velocity Composition

Consider a particle of fluid moving along a path.



Velocity Composition

Taylor Series Expansion of the velocity vector about the point P.



u - velocity component:

$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o + HOT$$

v - velocity component:

$$v = v_o + (x - x_o)\left(\frac{\partial v}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial v}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial v}{\partial z}\right)_o + HOT$$

w - velocity component:

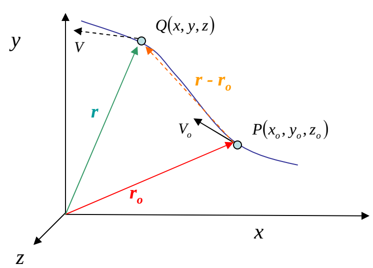
$$w = w_o + (x - x_o)\left(\frac{\partial w}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial w}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial w}{\partial z}\right)_o + HOT$$

If $(\mathbf{r} - \mathbf{r}_o)$ is small, we can neglect the HOT

Velocity Composition

Now a bit of Mathematical Jugglery:

1. Break up the partial differential terms

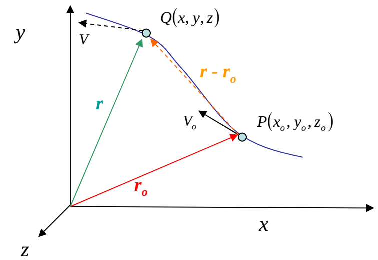


$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o$$

Velocity Composition

Now a bit of Mathematical Jugglery:

1. Break up the partial differential terms



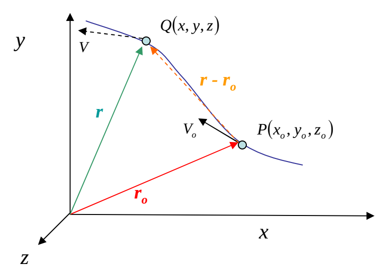
$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o$$

$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right]$$

Velocity Composition

Now a bit of Mathematical Jugglery:

1. Break up the partial differential terms
2. Add and subtract like differentials



$$u = u_o + (x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o$$

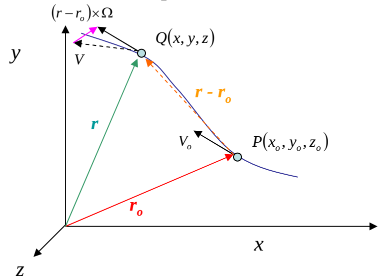
$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z}\right)_o\right]$$

$$u = u_o + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x} - \overbrace{\frac{\partial u}{\partial x}}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y} - \overbrace{\frac{\partial v}{\partial x}}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z} - \overbrace{\frac{\partial w}{\partial x}}\right)_o\right] + \frac{1}{2}\left[(x - x_o)\left(\frac{\partial u}{\partial x} + \overbrace{\frac{\partial u}{\partial x}}\right)_o + (y - y_o)\left(\frac{\partial u}{\partial y} + \overbrace{\frac{\partial v}{\partial x}}\right)_o + (z - z_o)\left(\frac{\partial u}{\partial z} + \overbrace{\frac{\partial w}{\partial x}}\right)_o\right]$$

Velocity Composition

We now have a new expression for the velocity vector V

For pure rotation & translation



$$u=u_o+\frac{1}{2}\left[(x-x_o)\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\right)_o+(y-y_o)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)_o+(z-z_o)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)_o\right]+\frac{1}{2}\left[(x-x_o)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right)_o+(y-y_o)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_o+(z-z_o)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)_o\right]$$

$$V=V_o+(r-r_o)\times\underbrace{\dot{\Omega}}+(r-r_o)\times\underbrace{\dot{S}}$$

Rotation Rate

Strain Rate – related to the deformation and dilatation of fluid elements

Velocity Composition

$$V=V_o+(r-r_o)\times\underbrace{\dot{\Omega}}+(r-r_o)\times\underbrace{\dot{S}}$$

Rotation Rate

Strain Rate

$$\dot{\Omega}=\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}$$

$$\dot{S}=\begin{bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \left(\frac{\partial v}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \left(\frac{\partial w}{\partial z}\right) \end{bmatrix}$$

Asymmetric

Symmetric

Since the stress tensor σ_{ij} is also symmetric, σ_{ij} can only be related to the Strain Rate

Velocity Composition

Deformation Rate Tensor

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Rotation Rate Tensor

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Asymmetric

Strain Rate Tensor

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Symmetric

$$e_{ij} = \Omega_{ij} + S_{ij}$$

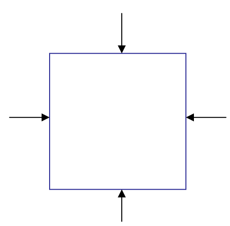
Constitutive Equations

Need to relate the strain rate tensor to the stress tensor:

1. If the fluid is a rest, the stress is hydrostatic and the pressure exerted on the fluid is thermodynamic pressure.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Shear Stress Tensor



2. Newtonian Fluid Approximation – shear stress is linearly related to the deformation-rate tensor.

3. Since there is no shearing action in a solid-body rotation of a fluid, no shear stresses will act during solid body rotation, but Ω_{ij} is non-zero in solid body rotation hence,

$\tau_{ij} \propto S_{ij} \longrightarrow$ Leads to 81 constants

Constitutive Equations

Need to relate the <u>strain rate tensor</u> to <u>the stress tensor:</u>

4. Fluid properties are isotropic

With these simplifications the constitutive relation for stress tensor becomes:

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \longrightarrow \text{2}^{\text{nd}}\text{ order symmetric tensor}$$

Where the viscous stress tensor is:

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

For incompressible fluids:

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

λ and μ are Viscosity Coefficients that must be determined empirically

Navier-Stokes Equations

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_j u_k}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}\left(-p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_j u_k}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left(\mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\right) + \rho f_j$$

$$\sigma_{ij} = -\left(p\delta_{ij} + \frac{2}{3}\mu\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Stokes Assumption

$$\lambda = -\frac{2}{3}\mu$$

For an incompressible fluid: $\frac{D\rho}{Dt} = 0$

$$\rho\frac{\partial u_j}{\partial t} + \rho u_k\frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \mu\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_i} + \rho f_j$$

Navier-Stokes Equations (incompressible flow)

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_i} + \rho f_j$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

4 Equations & 4 unknowns → u,v,w and P

We now have a closed system of Equations!!!