Runge-Kutta Methods – CH 25

Can achieve Taylor Series accuracy without evaluating higher order derivatives.

General form: $$y_{i+1} = y_i + \phi(x_i, y_i, h)h \qquad (1)$$

$\phi(x_i, y_i, h)$ - *Increment function* & is like a slope over the interval

$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$

- *a*'s are constants & *k*'s are recurrence relationships
- n=1 → Euler's method

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$$\phi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$
$$k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$
$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \ldots + q_{n-1,n-1} k_{n-1} h)$$

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Runge-Kutta Methods

To Determine the final form of (1)

1. Select n

2. Evaluate a's,p's,q's by setting the general form equal to terms in the T-S expansion.

3. For low-order forms
   - Number of terms n=order of the method
   - *Local truncation error* is $O(h^{n+1})$
   - *Global truncation error* is $O(h^n)$

See Box 25.1 in Text

2^{nd}- Order Runge-Kutta Methods

General Form: $$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h \qquad (2)$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

By setting (2) equal to a T-S expansion through the 2^{nd} order term, we can solve for a_1, a_2, p_1, q_{11}

$$\left.\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= 1/2 \\ a_2 q_{11} &= 1/2 \end{aligned}\right\}$$

3 Eqns & 4 unknows
Specify a_2 value →

$$\left\{\begin{aligned} a_1 &= 1 - a_2 \\ p_1 &= 1/(2a_2) \\ q_{11} &= 1/(2a_2) \end{aligned}\right.$$

**Since there are an infinite number of choices for a_2 there will be an infinite number of 2^{nd} order R-K Methods*

2^{nd}- Order Runge-Kutta Methods

A) Huen Method without iteration

$(a_2 = ½): a_1 = ½, p_1 = 1, q_{11} = 1$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

k_1 slope at start of interval
k_2 slope at end of interval

$$a_1 = 1 - a_2$$

$$p_1 = 1/(2a_2)$$

$$q_{11} = 1/(2a_2)$$

Global Truncation Error ~ $O(h^2)$

2^{nd}- Order Runge-Kutta Methods

B) Midpoint Method (a_2 = 1): $a_1 = 0$, $p_1 = 1/2$, q_{11} =1/2

$$y_{i+1} = y_i + k_2 h$$

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + 0.5h, y_i + 0.5k_1 h)$$

$$a_1 = 1 - a_2$$
$$p_1 = 1/(2a_2)$$
$$q_{11} = 1/(2a_2)$$

Global Truncation Error ~ $O(h^2)$

2^{nd}- Order Runge-Kutta Methods

C) Ralston's Method (a_2 = 2/3): $a_1 = 1/3$, $p_1 = 3/4$, q_{11} =3/4

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + 0.75h, y_i + .75k_1h)$$

$$a_1 = 1 - a_2$$

$$p_1 = 1/(2a_2)$$

$$q_{11} = 1/(2a_2)$$

Global Truncation Error ~ O(h^2)

4^{th} - Order Runge-Kutta Methods

Classical 4^{th} order RK Method – most commonly used RK method

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$y_{i+1} = y_i + \phi h$$

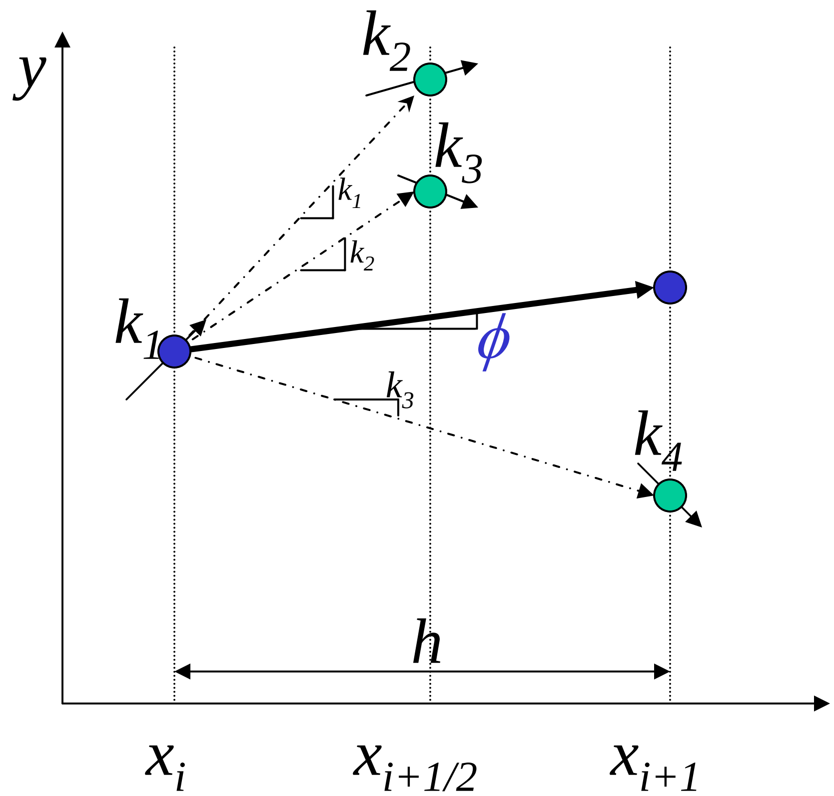
Slope Estimates:

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + 0.5h, y_i + .5k_1 h)$$

$$k_3 = f(x_i + 0.5h, y_i + .5k_2 h)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$



Global Truncation Error ~ $O(h^4)$

4th - Order Runge-Kutta Methods –

Example: Use classical RK4 to determine *y @ x=0.4* for *y'=x-y* and *h=0.4*

Recall the exact solution is:

$$y = x + e^{-x} - 1$$

$$y(0.4) = 0.070320$$

RK4 Solution:

$$\left.\begin{aligned} x_0 &= 0 \\ y_0 &= 0 \end{aligned}\right\} \text{Initial Conditions}$$

$$k_1 = f(x_i, y_i) = x_0 - y_0 = 0$$

$$k_2 = f(x_i + 0.5h, y_i + .5k_1h) = (0 + 0.4/2) - (0 + 0) = 0.2$$

$$k_3 = f(x_i + 0.5h, y_i + .5k_2h) = (0 + 0.4/2) - (0 + (0.5)(0.2)(0.4)) = 0.16$$

$$k_4 = f(x_i + 0.5h, y_i + k_3h) = (0 + 0.4) - (0 + 0.16(0.4)) = 0.336$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h = 0 + \frac{1}{6}(0 + 2(.2) + 2(.16) + .336)0.4$$

$$y_1 = 0.07040$$

4^{th} - Order Runge-Kutta Methods –

Example: Use classical RK4 to determine y @ *x=0.4* for *y'=x-y* and *h*=0.4

Error:

$$E_t = \left| \frac{.07032 - .07040}{.07032} \right| \bullet 100\% = .11\%$$

See Matlab Sample Matlab RK4 method

Method Comparison

- Higher order methods produce better accuracy
- Effort for the higher order methods is similar to low-order methods (much of the effort goes into evaluating the function)
- Classical 4^{th} order RK is most widely used as it produces accurate results with reasonable effort.

Systems of Equations

- Recall, Any n^{th} order ODE can be represented as a system of n 1st order ODEs

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

- To solve the system requires n initial conditions at $x = x_0$

Systems of Equations – RK4 Example

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2)$$

For example:

$$\frac{dy}{dx} = f_1(x, y, z) = -y$$

$$\frac{dz}{dx} = f_2(x, y, z) = 3 - 4\cos z + y$$

Subject to initial conditions

$$y_{1,0} = y_1(x = 0) = Y_1$$

$$y_{2,0} = y_2(x = 0) = Y_2$$

Systems of Equations – RK4 Example

Solve for slopes

$$k_{i,\,j}$$

*i*th value of k for the *j*th dependant variable

For RK-4 *i=1,2,3 and 4* while *j=1, 2,* … number of dependant variables

Systems of Equations – RK4 Example

Solve for slopes
Start with $i=0$
The initial condition

$$k_{1,1} = f_1\left(x_i, y_{1i}, y_{2i}\right)$$

$$k_{1,2} = f_2\left(x_i, y_{1i}, y_{2i}\right)$$

$$k_{2,1} = f_1\left(x_i + \frac{1}{2}h, y_{1i} + \frac{1}{2}k_{11}h, y_{2i} + \frac{1}{2}k_{12}\right)$$

$$k_{2,2} = f_2\left(x_i + \frac{1}{2}h, y_{1i} + \frac{1}{2}k_{11}h, y_{2i} + \frac{1}{2}k_{12}h\right)$$

$$k_{3,1} = f_1\left(x_i + \frac{1}{2}h, y_{1i} + \frac{1}{2}k_{21}h, y_{2i} + \frac{1}{2}k_{22}h\right)$$

$$k_{3,2} = f_2\left(x_i + \frac{1}{2}h, y_{1i} + \frac{1}{2}k_{21}h, y_{2i} + \frac{1}{2}k_{22}h\right)$$

$$k_{4,1} = f_1\left(x_i + h, y_{1i} + k_{31}h, y_{2i} + k_{32}h\right)$$

$$k_{4,2} = f_2\left(x_i + h, y_{1i} + k_{31}h, y_{2i} + k_{32}h\right)$$

Systems of Equations – RK4 Example

$$y_{1,i+1} = y_{1i} + \frac{1}{6}\left(k_{11} + 2k_{21} + 2k_{31} + k_{41}\right)h$$

$$y_{2,i+1} = y_{2i} + \frac{1}{6}\left(k_{12} + 2k_{22} + 2k_{32} + k_{42}\right)h$$

Show Matlab Systems of Equations RK4 Example

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = -\frac{y_2}{2} - 7y_1$$

$$y_1\left(x = 0\right) = 4$$

$$y_2\left(x = 0\right) = 0$$

Matlab ODE solvers

ODE23 and ODE45 are RK solvers that combine 2nd and 3rd order RK and 4th and 5th order RK methods.

See Chapter 8 in Palm Text.