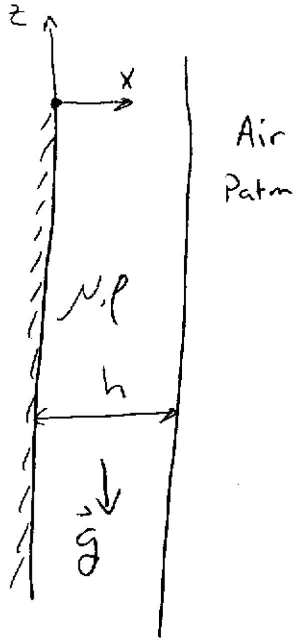
NS - EXAMPLE: FLOW OF A THIN FILM DOWN A VERTICAL INFINATELY WIDE WALL. NO PRESSURE FORCE DRIVING THE FLOW, ONLY GRAVITY.



* CALCULATE THE VELOCITY IN THE FILM

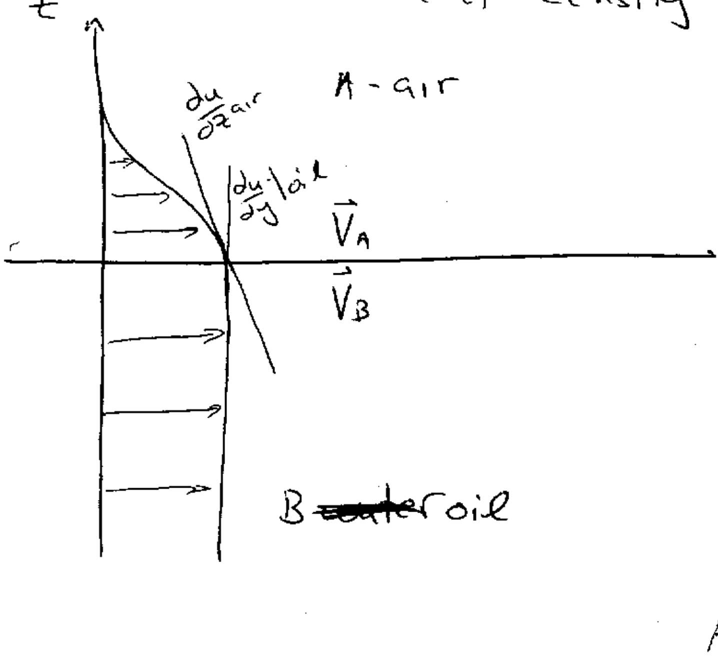
Assumptions

1. Steady
2. Incompressible
3. parallel ($u = 0$, everywhere)
4. Laminar
5. Newtonian
6. $P = P_{atm}$ Everywhere
7. 2D flow ($v = 0$)
8. gravity acts as $\vec{g} = -g\hat{k}$

Boundary Conditions

1. No Slip Condition, $u = v = w = 0$ @ $x = 0$
2. Negligable Shear stress at the free surface
   $\frac{dw}{dx} = 0$ @ $x = h$

ASIDE: Consider two fluids with substantially different density & viscosity (air & water)



At the interface

$\vec{V}_A = \vec{V}_B$

$\vec{\tau}_{s,A} = \vec{\tau}_{s,B}$

$$\tau_{s,oil} = \mu_{oil} \frac{du}{dz} = \mu_{Air} \frac{du}{dz}$$

$$\mu_{oil} >> 100 \mu_{air}$$

$$\frac{\partial u}{\partial z}\Big|_{oil} << \frac{du}{dz}\Big|_{air}$$

At boundary

$\tau_{liquid} \approx 0$, $P_{liq} = P_{air}$

* Incompressible Continuity Equation:

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{assump. 3}} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{assump. 7}} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial z} = 0 \quad \text{Fully Developed}$$

hence $w = w(x)$

z-direction NS

$$\rho\left(\underbrace{\cancel{\frac{\partial w}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{u\frac{\partial w}{\partial x}}}_{3} + \underbrace{\cancel{v\frac{\partial w}{\partial y}}}_{7} + \underbrace{\cancel{w\frac{\partial w}{\partial z}}}_{\text{continuity}}\right) = \underbrace{-\cancel{\frac{\partial p}{\partial z}}}_{6} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 w}{\partial y^2}}}_{\text{2D flow}} + \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{\text{continuity}}\right) + \underbrace{\rho g_z}_{-\rho g}$$

$$\frac{d^2 w}{dx^2} = \frac{\rho g}{\mu}$$

$$\frac{dw}{dx} = \frac{\rho g}{\mu} x + C_1$$

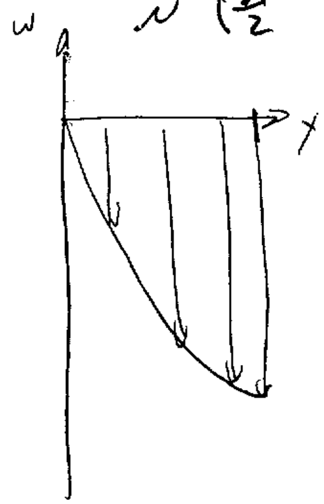
$$w = \frac{\rho g}{\mu}\frac{x^2}{2} + C_1 x + C_2$$

Apply B.C.s

① $w(0) = 0 \Rightarrow C_2 = 0$

② $\frac{dw}{dz}(h) = 0 \Rightarrow C_1 = -\frac{\rho g h}{\mu}$

$$w = \frac{\rho g}{\mu}\left(\frac{x^2}{2} - h x\right)$$



at $x = h \Rightarrow w = -\frac{\rho g}{\mu}\frac{h^2}{2}$