1. Incompressible N-S & continuity Eqns, laminar flow
- 2D so, $w = 0$, No press grad in horiz
fully developed (f.D.)

a. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$

$$\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{S.S.} + u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{f.D.} + v \underbrace{\cancel{\frac{\partial u}{\partial y}}}_{(*)} + w \underbrace{\cancel{\frac{\partial u}{\partial z}}}_{2D} = -\frac{1}{\rho}\cancel{\frac{\partial p}{\partial x}} + \nu\left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{f.D.} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 u}{\partial z^2}}}_{2D}\right)$$

$$\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{S.S.} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\underbrace{\cancel{\frac{\partial v}{\partial z}}}_{2D} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$u(x,0) = v(x,0) = 0$

$u(x,h) = U$, $v(x,h) = 0$

$\underbrace{\frac{\partial u}{\partial x}}_{f.D.} + \frac{\partial v}{\partial y} = 0 \Rightarrow \therefore$ since $v(x,0) = 0$ $v(x,y) = 0$ (*)

$$0 = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow \text{Integrate Twice}$$

$$u(y) = Ay + B$$

$u(0) = 0 \Rightarrow B = 0$

$u(h) = U \rightarrow A = \frac{U}{h}$

$$\boxed{u(y) = \frac{U}{h}y}$$

5

(b) (1) Strain Rate tensor $\Rightarrow S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$S_{ij} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{U}{2h} & 0 \\ \frac{U}{2h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

Since only $\frac{\partial u}{\partial y} \neq 0$ so

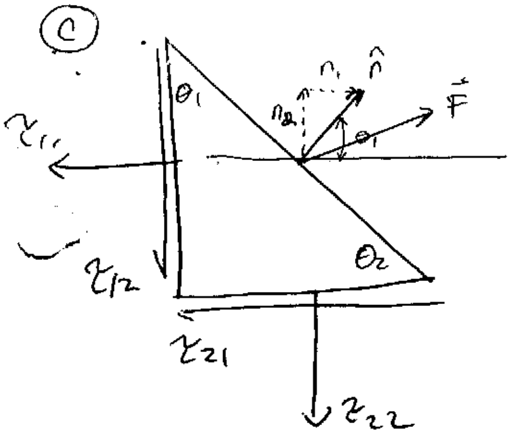
$S_{12} = \frac{1}{2}\frac{\partial u_1}{\partial x_2} = \frac{1}{2}\frac{\partial u}{\partial y} = \frac{U}{2h}$

(2) Stress tensor $\Rightarrow \sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (Incompressible flow so $\frac{\partial u_i}{\partial x_i} = 0$)

$$\sigma_{ij} = \begin{pmatrix} -p & \mu\frac{U}{h} & 0 \\ \mu\frac{U}{h} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

3

(c)



$\vec{F} = f_i$

$$\tau_{ij} = \begin{bmatrix} 0 & \mu \frac{U}{h} \\ \mu \frac{U}{h} & 0 \end{bmatrix}$$

Force per unit Area:

$$f_i = n_j \tau_{ji}$$

$$f_1 = n_1 \cancel{\tau_{11}}^{0} + n_2 \tau_{21} \qquad \text{but } \tau_{11} = \tau_{22} = 0$$

$$f_2 = n_1 \tau_{12} + n_2 \cancel{\tau_{22}}_{0}$$

$$n_1 = \cos\theta_1 = \cos(50^\circ) = 0.6428$$

$$n_2 = \cos\theta_2 = \cos(40^\circ) = 0.766$$

$$f_1 = n_2 \tau_{21} = 0.766 \mu \frac{U}{h} \quad \Rightarrow \quad |f| = (f_1^2 + f_2^2)^{1/2} = \mu \frac{U}{h}$$

$$f_2 = n_1 \tau_{12} = 0.6428 \mu \frac{U}{h}$$

3

$$\left.\begin{aligned} \sin\phi &= \frac{f_2}{|f|} = 0.6428 \\ \cos\phi &= \frac{f_1}{|f|} = 0.766 \end{aligned}\right\} \phi = 40^\circ$$

(d) <u>Viscous</u> <u>Dissipation</u>, Φ for an incompressible flow

$$\phi = 2\mu S_{ij} S_{ij} = \frac{\mu}{2} \left(\frac{du_i}{dx_j} + \frac{du_j}{dx_i} \right)^2$$

$$= 2\mu \left(\cancel{S_{11}}^{0} \cancel{S_{11}} + S_{21}^2 + \cancel{S_{31}^2}^{0} + S_{12}^2 + \cancel{S_{22}^2}^{0} + \cancel{S_{32}^2}^{0} + \cancel{S_{13}^2}^{0} + \cancel{S_{23}^2}^{0} + \cancel{S_{33}^2}^{0} \right)$$

$$= 2\mu \left(S_{21}^2 + S_{12}^2 \right) = 2\mu \left(\frac{U^2}{4h^2} + \frac{U^2}{4h^2} \right)$$

$$\boxed{\phi = \mu \frac{U^2}{h^2}}$$

3

e

$$\underbrace{\int_{cv} \frac{dE}{dt}\, d\forall}_{A} + \underbrace{\int_{cs} E(\vec{V}\cdot\hat{n})\, dA}_{B} = \underbrace{\int_{cs} \rho(\vec{g}\cdot\vec{v})\, d\forall}_{C} + \underbrace{\int_{cs} u_j \sigma_{ij} n_i\, dA}_{D} + \underbrace{\int_{v} p(\vec{\nabla}\cdot\vec{v})\, d\forall}_{E} - \underbrace{\int_{cv} \phi\, d\forall}_{F}$$

where $E = \rho \frac{u_i^2}{2} \Rightarrow$ Kinetic Energy per unit volume

A $= 0 \Rightarrow$ steady flow

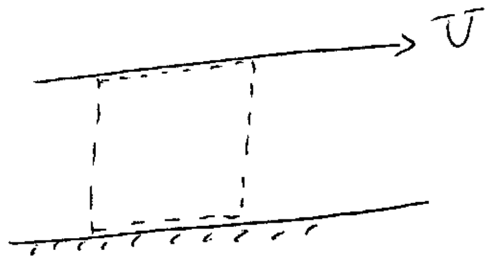
B $= 0 \Rightarrow$ fully developed

C $= 0 \Rightarrow$ Neglect hydrostatic

D $\Rightarrow$ Include

E $\Rightarrow 0 \Rightarrow$ Incompressible flow

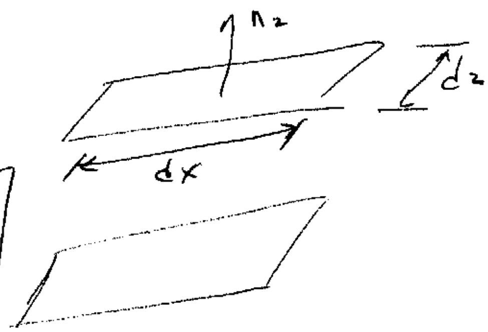
F $\Rightarrow$ Include



$$\int_{cs} u_j \sigma_{ij} n_i\, dA = \int_{cv} \phi\, d\forall$$

Pressure terms cancel out so we are left with

$$\int_{cs} U_1 \sigma_{z1} n_z\, dx\, dz = \int_{cv} \mu \frac{U^2}{h^2}\, dx\, dy\, dz$$



$$\boxed{\iint U\left(\mu \frac{U}{h}\right) dx\, dy = \iint \mu \frac{U^2}{h}\, dx\, dz}$$

f. $-\frac{\partial P}{\partial x} = a$, couette flow with axial Pressure gradient

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x} = a$$

3

$$\frac{d^2 u}{dy^2} = \frac{a}{\nu}$$

$$\frac{du}{dy} = \frac{a}{\nu} y + B$$

$$u = \frac{a}{\nu}\frac{y^2}{2} + By + c$$

boundary conditions

$u(y=0) = 0$ (1)

$u(y=h) = \bar{U}$ (2)

B.C. 1 $\rightarrow$ $C = 0$

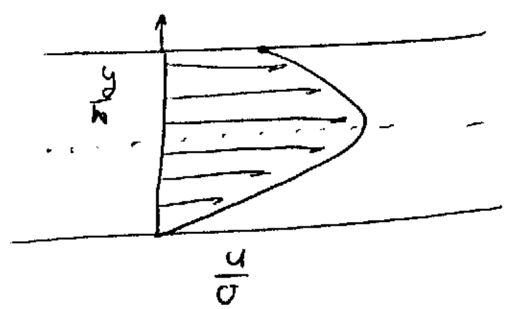
B.C. 2

$$\bar{U} = \frac{a}{\nu}\frac{h^2}{2} + Bh$$

$$B = \frac{\bar{U}}{h} - \frac{a}{\nu}\frac{h}{2}$$

$$u = \frac{a}{\nu}\frac{y^2}{2} + \left(\frac{\bar{U}}{h} - \frac{a}{\nu}\frac{h}{2}\right)y$$

$$\boxed{u(y) = \frac{a}{2\nu}(y^2 - yh) + \frac{\bar{U} y}{h}}$$



(g) Just like problem c except now $\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is

$$\tau_{ij} = \begin{bmatrix} 0 & \mu\frac{du}{dy} & 0 \\ \mu\frac{du}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{du}{dy} = \frac{a}{2\nu}(2y - h) + \frac{U}{h}$$

Since $n_1 = \frac{1}{2}$, $n_2 = 0.866$ are unchange

$$f_1 = \overset{0.766}{0.866}\left(\frac{a}{2}(2y-h) + \frac{U\nu}{h}\right)$$

$$f_2 = \overset{.6428}{\frac{1}{2}}\left(\frac{a}{2}(2y-h) + \frac{U\nu}{h}\right)$$

$$f = (f_1^2 + f_2^2)^{1/2}$$

$$\sin\phi = \frac{f_2}{f}$$

$$\cos\phi = \frac{f_1}{f}$$

Three-Dimensional Incompressible Flow

Four variables or unknowns:

- Pressure P
- Three components of velocity $\vec{V}$

Four equations of motion:

- Continuity,
  $\vec{\nabla} \cdot \vec{V} = 0$
- Three components of Navier–Stokes,
  $\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$

FIGURE 9–43
A general three-dimensional but incompressible flow field with constant properties requires four equations to solve for four unknowns.

9–6 ▪ DIFFERENTIAL ANALYSIS OF FLUID FLOW PROBLEMS

In this section we show how to apply the differential equations of motion in both Cartesian and cylindrical coordinates. There are two types of problems for which the differential equations (continuity and Navier–Stokes) are useful:

- Calculating the pressure field for a known velocity field
- Calculating both the velocity and pressure fields for a flow of known geometry and known boundary conditions

For simplicity, we consider only incompressible flow, eliminating calculation of ρ as a variable. In addition, the form of the Navier–Stokes equation derived in Section 9–5 is valid only for Newtonian fluids with constant properties (viscosity, thermal conductivity, etc.). Finally, we assume negligible temperature variations, so that T is not a variable. We are left with four variables or unknowns (pressure plus three components of velocity), and we have four differential equations (Fig. 9–43).

Calculation of the Pressure Field for a Known Velocity Field

The first set of examples involves calculation of the pressure field for a known velocity field. Since pressure does not appear in the continuity equation, we can theoretically generate a velocity field based solely on conservation of mass. However, since velocity appears in both the continuity equation and the Navier–Stokes equation, these two equations are *coupled*. In addition, pressure appears in all three components of the Navier–Stokes equation, and thus the velocity and pressure fields are also coupled. This intimate coupling between velocity and pressure enables us to calculate the pressure field for a known velocity field.

Problem #2

EXAMPLE 9–13 **Calculating the Pressure Field in Cartesian Coordinates**

Consider the steady, two-dimensional, incompressible velocity field of Example 9–9, namely, $\vec{V} = (u, v) = (ax + b)\vec{i} + (-ay + cx)\vec{j}$. Calculate the pressure as a function of x and y.

SOLUTION For a given velocity field, we are to calculate the pressure field.

Assumptions 1 The flow is steady. 2 The fluid is incompressible with constant properties. 3 The flow is two-dimensional in the xy-plane. 4 Gravity does not act in either the x- or y-direction.

Analysis First we check whether the given velocity field satisfies the two-dimensional, incompressible continuity equation:

$$\underbrace{\frac{\partial u}{\partial x}}_{a} + \underbrace{\frac{\partial v}{\partial y}}_{-a} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0\text{ (2-D)}} = a - a = 0$$

Thus, contin
were *not* sat
not be physi
Next, we c

$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{0\text{ (steady)}} + \ldots\right.$

The y-mome

The y-mome
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reduces to

The x-mome
pressure fiel
In order fo
Furthermore
a pressure fi
sudden disc
requires that
matter (Fig.
2 and 3, res

$\frac{\partial^2 P}{\partial x\, \partial y}$

Equation 4
velocity fiel
Stokes equa
If at this p
yield two in
9–44 were
could not sa
equation, ar
field.
To calcula
obtain an ex

Pressure fiel

Note that w
constant of
partial deriv

...ntinuity is indeed satisfied by the given velocity field. If continuity ... satisfied, we would stop our analysis—the given velocity field would ... physically possible, and we could not calculate a pressure field.

... we consider the y-component of the Navier–Stokes equation:

$$\rho\left(\cdots + \underbrace{u\frac{\partial v}{\partial x}}_{(at+b)c} + \underbrace{v\frac{\partial v}{\partial y}}_{(-ay+cx)(-a)} + \underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{0\,(2\text{-D})}\right) = -\frac{\partial P}{\partial y} + \underbrace{\cancel{\rho g_y}}_{0} + \mu\left(\underbrace{\cancel{\frac{\partial^2 v}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v}{\partial y^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{0\,(2\text{-D})}\right)$$

... momentum equation reduces to

$$\frac{\partial P}{\partial y} = \rho(-acx - bc - a^2y + acx) = \rho(-bc - a^2y) \qquad \textbf{(2)}$$

... momentum equation is satisfied, provided we can generate a pressure ... that satisfies Eq. 2. In similar fashion, the x-momentum equation ...es to

$$\frac{\partial P}{\partial x} = \rho(-a^2x - ab) \qquad \textbf{(3)}$$

... momentum equation is also satisfied, provided we can generate a ... field that satisfies Eq. 3.

... for a steady flow solution to exist, P cannot be a function of time. ...more, a physically realistic steady, incompressible flow field requires ... field $P(x, y)$ that is a smooth function of x and y (there can be no ... discontinuities in either P or a derivative of P). Mathematically, this ... that the order of differentiation (x then y versus y then x) should not ... (Fig. 9–44). We check whether this is so by cross-differentiating Eqs. ... 3, respectively,

$$\frac{\partial^2 P}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) = 0 \quad \text{and} \quad \frac{\partial^2 P}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) = 0 \qquad \textbf{(4)}$$

...ion 4 shows that P is a smooth function of x and y. Thus, *the given ... field satisfies the steady, two-dimensional, incompressible Navier–... equation.*

... this point in the analysis, the cross-differentiation of pressure were to ... two incompatible relationships, in other words if the equation in Fig. ... were not satisfied, we would conclude that the given velocity field ... not satisfy the steady, two-dimensional, incompressible Navier–Stokes ...tion, and we would abandon our attempt to calculate a steady pressure ...

... calculate $P(x, y)$, we partially integrate Eq. 2 (with respect to y) to ... an expression for $P(x, y)$,

...re field from y-momentum:

$$P(x, y) = \rho\left(-bcy - \frac{a^2y^2}{2}\right) + g(x) \qquad \textbf{(5)}$$

... that we add an arbitrary function of the other variable x rather than a ...ant of integration since this is a partial integration. We then take the ... derivative of Eq. 5 with respect to x to obtain

$$\frac{\partial P}{\partial x} = g'(x) = \rho(-a^2x - ab) \qquad \textbf{(6)}$$

Cross-Differentiation, xy-Plane

$P(x, y)$ is a smooth function of x and y only if the order of differentiation does not matter:

$$\frac{\partial^2 P}{\partial x\,\partial y} = \frac{\partial^2 P}{\partial y\,\partial x}$$

FIGURE 9–44
For a two-dimensional flow field in the xy-plane, cross-differentiation reveals whether pressure P is a smooth function.

where we have equated our result to Eq. 3 for consistency. We now integrate Eq. 6 to obtain the function $g(x)$:

$$g(x) = \rho\left(-\frac{a^2x^2}{2} - abx\right) + C_1 \quad (7)$$

where C_1 is an arbitrary constant of integration. Finally, we substitute Eq. 7 into Eq. 5 to obtain our final expression for $P(x, y)$. The result is

$$P(x,y) = \rho\left(-\frac{a^2x^2}{2} - \frac{a^2y^2}{2} - abx - bcy\right) + C_1 \quad (8)$$

Discussion For practice, and as a check of our algebra, you should differentiate Eq. 8 with respect to both y and x, and compare to Eqs. 2 and 3. In addition, try to obtain Eq. 8 by starting with Eq. 3 rather than Eq. 2; you should get the same answer.

Notice that the final equation (Eq. 8) for pressure in Example 9–13 contains an arbitrary constant C_1. This illustrates an important point about the pressure field in an incompressible flow; namely,

> The velocity field in an incompressible flow is not affected by the absolute magnitude of pressure, but only by pressure differences.

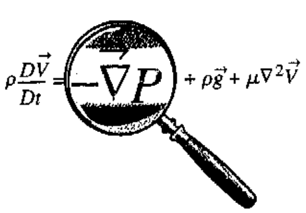


FIGURE 9–45
Since pressure appears only as a gradient in the incompressible Navier–Stokes equation, the absolute magnitude of pressure is not relevant—only pressure *differences* matter.

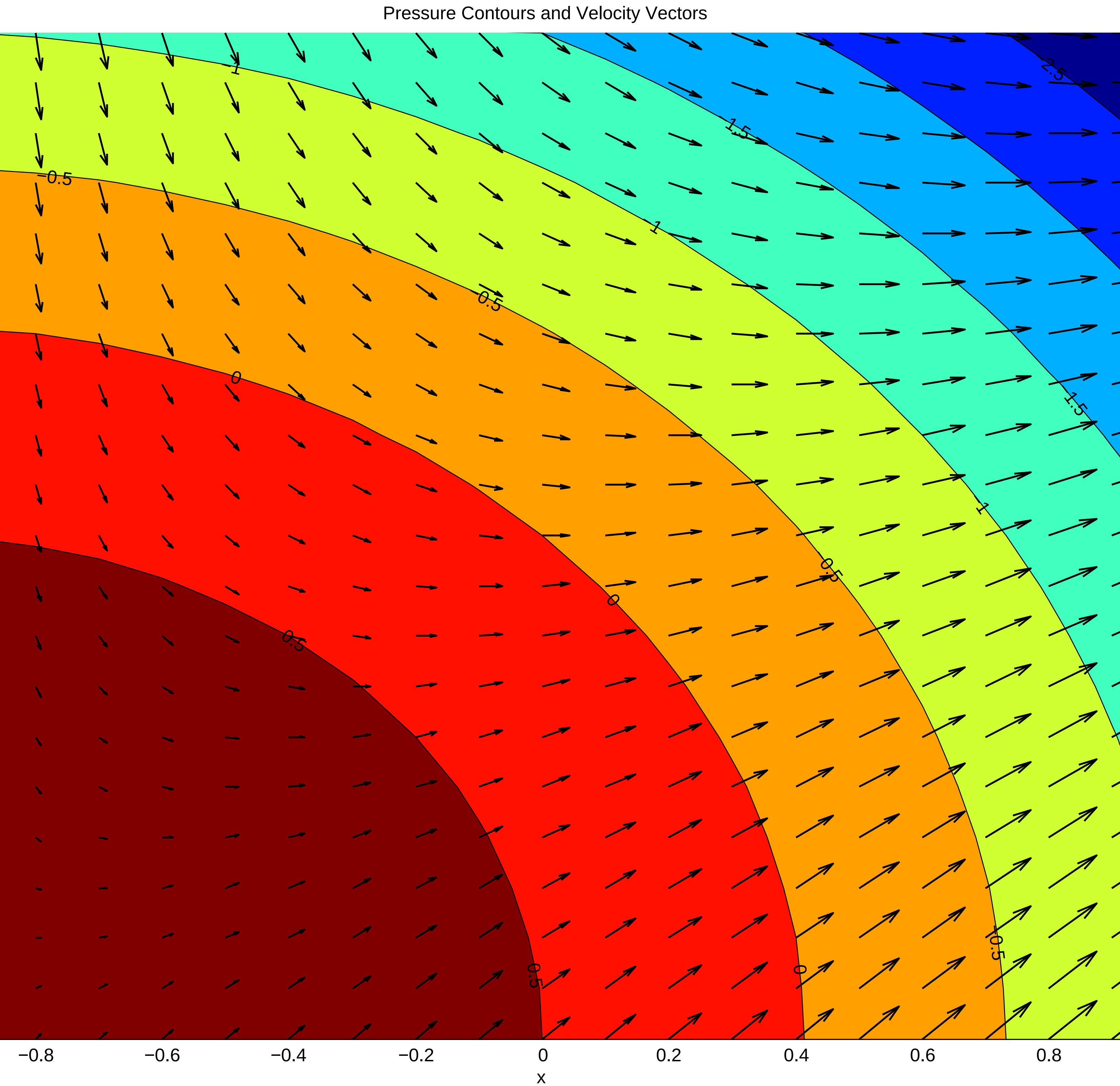
This should not be surprising if we look at the Navier–Stokes equation, where P appears only as a *gradient*, never by itself. Another way to explain this statement is that it is not the absolute magnitude of pressure that matters, but only pressure *differences* (Fig. 9–45). A direct result of the statement is that we can calculate the pressure field to within an arbitrary constant, but in order to determine that constant (C_1 in Example 9–13), we must measure (or otherwise obtain) P somewhere in the flow field. In other words, we require a pressure boundary condition.

We illustrate this point with an example generated using **computational fluid dynamics (CFD),** where the continuity and Navier–Stokes equations are solved numerically (Chap. 15). Consider downward flow of air through a channel in which there is a nonsymmetrical blockage (Fig. 9–46). (Note that the computational flow domain extends much further upstream and downstream than shown in Fig. 9–46.) We calculate two cases that are identical except for the pressure condition. In case 1 we set the gage pressure far downstream of the blockage to zero. In case 2 we set the pressure at the same location to 500 Pa gage pressure. The gage pressure at the top center of the field of view and at the bottom center of the field of view are shown in Fig. 9–46 for both cases, as generated by the two CFD solutions. You can see that the pressure field for case 2 is identical to that of case 1 except that the pressure is everywhere increased by 500 Pa. Also shown in Fig. 9–46 are a velocity vector plot and a streamline plot for each case. The results are identical, confirming our statement that the velocity field is not affected by the absolute magnitude of the pressure, but only by pressure *differences*. Subtracting the pressure at the bottom from that at the top, we see that ΔP = 12.784 Pa for both cases.

The statement about pressure differences is *not* true for *compressible* flow fields, where P is the thermodynamic pressure rather than the mechanical pressure. In such cases, P is coupled with density and temperature through

```
%Homework 2, problem 2
clear;
clc;
rho = 1 %kg/m^3
C1 = 0  %Pa
[x,y] = meshgrid(-1:.1:1,-1:.1:1);
a = 1;
b = 1;
c =1;
%calculate u-velocity
u=a*x + b;
%calculate v-velocity
v = -a*y+ c*x;


%Now plot pressure contours
P = rho*(-(a^2*x.^2)/2 - (a^2*y.^2)/2 -a*b*x - b*c*y) + C1;
hold
contourf(x,y,P)
%create velocity vector plot
quiver(x,y,u,v,'-k')
colorbar
```


Pressure Contours and Velocity Vectors
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
x
−2.5
−1.5
−1
−0.5
0
0.5