Introduction to Vorticity

ME 6700 – Intermediate Fluid Dynamics

Vorticity Vector – $\vec{\omega}$

1. Vector quantity that is proportional to the angular momentum of a fluid element.
2. The vorticity vector is defined by the flow field. Thus

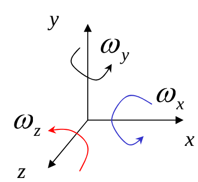
$$\vec{\omega} = \vec{\omega}(x, y, z, t)$$

$$\vec{\omega} = \vec{\nabla} \times \vec{V}$$

$$\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Right Hand Rule Convention →



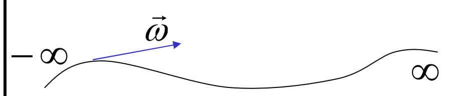
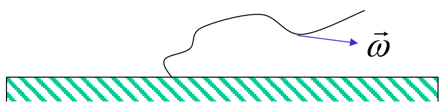
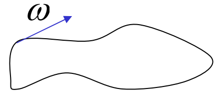
Vorticity – Properties

1. Vortex lines are everywhere tangent to the vorticity vector.
2. The vorticity field is solenoidal. That is, the divergence of the curl of a vector is identically zero. Thus,

$$\vec{\nabla} \bullet \vec{\omega} = \vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{V} \right) = 0$$

Clear analogy with *conservation of mass* and *streamlines*

$$\vec{\nabla} \bullet \vec{\omega} = 0$$

	1. infinite extent
	2. One end attached to a wall (the other may extend to infinity or attach to another wall)
	3. Continuous loop

Vorticity Vector – $\vec{\omega}$

Vorticity is directly related to the rotation tensor $\dot{\Omega}$

$$\dot{\Omega} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & \omega_y \\ -\omega_z & 0 & \omega_x \\ -\omega_y & -\omega_x & 0 \end{bmatrix}$$

Skew-symmetric or asymmetric Tensor that accounts for rigid body rotation

Vorticity in the Incompressible Navier-Stokes Equations

The vorticity vector is defined by the flow field. Thus

$$\rho \frac{D\vec{V}}{Dt} = \rho\left(\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \vec{\nabla}\right)\vec{V}\right) = -\vec{\nabla} P + \mu \nabla^2 \vec{V} \quad (1)$$

$$\left(\vec{\nabla} \bullet \vec{V}\right) = 0$$

Using the following vector identities:

$$\left(\vec{V} \bullet \vec{\nabla}\right)\vec{V} = \frac{1}{2}\vec{\nabla}\left(\vec{V} \bullet \vec{V}\right) - \vec{V} \times \left(\vec{\nabla} \times \vec{V}\right)$$

$$\nabla^2 \vec{V} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) - \vec{\nabla}\left(\vec{\nabla} \bullet \vec{V}\right)$$

We can rewrite (1) as:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\vec{\nabla}\left(V^2\right)\right) + \vec{\nabla} P = \left(\vec{V} \times \vec{\omega}\right) + \mu\left(\vec{\nabla} \times \vec{\omega}\right)$$

Irrotational Flow $\vec{\omega} = 0$

For an irrotational flow, the vorticity vanishes and the flow is described by the Euler equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P$$

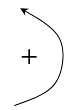
Note: we have not forced $\mu = 0$

Thus, We often chose to use vorticity as a variable when viscosity is important.

- Shear Layers
- Boundary Layers

Circulation

- Represents the “swirl” of the flow
- The net flux of vorticity through a surface area bounded by a closed path



$$\Gamma = \oint_C \vec{V} \bullet d\vec{l} = \int_A \vec{\omega} \bullet \hat{n} dA$$